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A New Critical Value Concerning the Genealogy of Long Period Families at L_4 in the Restricted Three-body Problem *

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Abstract We found another critical mass ratio value $\overline{\mu}$ between μ_4 and μ_5 concerning the genealogy of the long period family around the equilateral equilibrium point L_4 in the restricted three-body problem. This value has not been pointed out before. We used numerical computations to show how the long period family evolves around this critical value. The case is similar to that of the critical values between μ_2 and μ_4, with slight difference in evolution details.

Key words: celestial mechanics — restricted three-body problem — equilateral equilibrium point — periodic orbits

1 INTRODUCTION

As is well known, when the mass ratio μ is smaller than $\mu_1 = 0.0385208965$, there exist two families of periodic orbits—the short period family and the long period family (Szebehely 1967). According to Lyapunov's theorem, there are exceptions to this conclusion when the frequencies of the short and long period families are of commensurability k (k an integer). The mass ratios for commensurability to happen are

$$\mu_k = [1 - (k^4 + 38k^2/27 + 1)^{1/2}/(k^2 + 1)]/2\,. \tag{1}$$

There have been a few papers concerning the global evolution of these two families (Deprit & Henrard 1968; Henrard 2002). From these studies, we know that the short period family terminates continuously onto a Lyapunov planar orbit emanating from the collinear libration point L_3 for every μ between 0 and μ_1, but the evolution of the long period family depends on the parameter μ. Generally, when μ is between μ_k and μ_{k+1}, the long period family terminates onto a short period orbit travelling $(k + 1)$ times. We can consider the long period family as a $bridge$ connecting the equilibrium point L_4 and a short period orbit travelling $(k+1)$ times, so we can also denote the long period family as $B(L_4, (k+1)S)$. From this short period orbit emanates another bridge connecting another short period orbit travelling $(k+2)$ times. The process continues with more and more bridges $B((k + l)S, (k + l + 1)S), (l \geq 2)$ until it reaches a homoclinic orbit emanating from L_4 (Henrard 1983). Besides these bridges, there exists another double-lane bridge $B(kS, kS')$ connecting two different short period orbits travelling k times. When μ grows to be between μ_{k-1} and μ_k, the double-lane bridge $B(kS, kS')$ breaks into $B(L_4, kS)$ and $B(kS, (k + 1)S)$ while annexing another double-lane bridge $B((k - 1)S, (k - 1)S')$.

However, Henrard (1970) found that this scenario breaks down when k reaches 3. Between μ_3 and μ_4 there stands a special value μ^*. One lane of the bridge $B(3S, 3S')$ breaks when μ is larger than μ^*, but before it reaches μ_3. The other lane of the bridge $B(3S, 3S')$ survives until μ reaches μ_3. A similar

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scenario happens to $B(2S, 2S')$ with another special value μ^{**} between μ_2 and μ_3. Henrard (1970) thought that there might exist other critical values between μ_k and μ_{k+1} for $k > 3$, but in his subsequent papers (Henrard 2002) he did not give any other critical values. In our computation of these bridges, we found that there exists another critical value between μ_4 and μ_5 whose role is very similar to that of μ^* and μ^{**}. Denoting this value as $\overline{\mu}$, we used numerical computations to show how the long family evolves around this critical value.

2 METHODOLOGY

We used numerical methods to compute the evolution of the short and long period families, and the bridges connecting them. Denote the equation of motion of the planar restricted three-body problem by

$$\dot{X} = F(X), \tag{2}$$

where $X = (x, y, \dot{x}, \dot{y})$ is a four-dimensional vector in a chosen coordinate representing the state vector of the massless small body. In our computations, the coordinate chosen is not the usual synodic one for the restricted three-body problem, rather, it is one with origin at L_4 and with a fixed rotation angle from the synodic coordinate. The details can be found in the book of Szehebely (Szebehely 1967). Given the initial state vector X_0 at time t_0, integration of Equation (2) gives an orbit of the system. Denoting the orbit as $X(X_0, t_0 + t)$, a periodic orbit with period T satisfies the following condition,

$$X(X_0, t_0 + T) - X_0 = 0. \tag{3}$$

Given initial guesses for X_0 and T, Equation (3) can be solved by iteration. The iteration equation is

$$(\partial X/\partial X_0 - I) \cdot \delta X_0 + \partial X/\partial T \cdot \delta T = 0, \tag{4}$$

where $M = \partial X/\partial X_0$ is the monodromy matrix. For autonomous dynamical systems such as Equation (2), if X_0 is a solution of Equation (3), then $X(X_0, t_0 + t), \forall t \in \mathbf{R}$ is also a solution of Equation (3), and since the periodic orbits are generally embedded inside a family, there exists different X_0 for different period T. Moreover, only three of the four equations in Equation (3) are independent because of the existence of the energy equation. So the solution of Equation (3) is not unique. Denoting $\{(X_0, T)|X(X_0, t_0+T)-X_0 = 0\}$ as the solution space, it is a subset in $\mathbf{R}^5$. Equation (3) cannot be solved unless we fix the period T and one component of X_0 (for example, we could fix $y \equiv 0$ and leave $x_0, \dot{x_0}, \dot{y_0}$ variable in the iteration).

If we know one periodic orbit in a family, we can use numerical continuation to compute the other members of the family using Equation (4). The algorithm used is the so-called predictor-corrector algorithm. For example, if we keep $y \equiv 0$, we have $\{(x_0, \dot{x_0}, \dot{y_0}, T)|X(X_0, t_0 + T) - X_0 = 0\}$ as the solution space. First we denote the already known periodic orbit as $(x'_0, \dot{x}'_0, \dot{y}'_0, T')$, then we change the period from T' to $T' + \delta T$, and then the other three variables $x_0, \dot{x_0}, \dot{y_0}$ will change according to Equation (4). We denote these changes as $\delta x_0, \delta \dot{x_0}, \delta \dot{y_0}$. This step is called the predictor phase. Next, we use the changed values $(x'_0 + \delta x_0, \dot{x}'_0 + \delta \dot{x_0}, \dot{y}'_0 + \delta \dot{y_0}, T' + \delta T)$ as the initial values for Equation (3) and keep the period fixed, then we obtain a periodic orbit of period $T' + \delta T$ after the iteration. This is the corrector phase. This predictor-corrector algorithm is widely used in numerical analysis. The fixed parameter in the corrector phase is called the parameter of the family. It can be any one of the four components of X_0 or the period T. In our continuations, we used x_0 as the parameter of the family. When it reaches an extreme value x_0, we change the parameter from x_0 to the period T.

Since Equation (2) describes an autonomous Hamiltonian system, the eigenvalues of the monodromy matrix are of the form $(1, 1, \lambda, 1/\lambda)$ (Arnold 1999). When $|\lambda + 1/\lambda|$ is greater than 2, the periodic orbit is unstable. When $|\lambda + 1/\lambda|$ is smaller than 2, the periodic orbit is stable. When $|\lambda + 1/\lambda|$ equals 2, we have the critical case. The periodic orbit can be stable or unstable depending on the nonlinear effects. The most special case appears when $\lambda = 1$. The eigenvector corresponding to eigenvalue '1' is the direction along which a displacement of the state vector preserves the periodicity. So if $\lambda = 1$, apart from the original direction, there may exist one or two other directions along which a displacement of the state vector preserves the periodicity. However, this is not necessarily always the case: it depends on the geometric multiplicity of the monodromy matrix (Henrard 2002), and this only occurs when the geometric multiplicity is larger than 1. From the new direction other families of periodic orbits may be emanate. This phenomenon is called bifurcation. A more general form of bifurcation is the so-called n-bifurcation (Henrard 2002). It is the case when $\lambda = \exp(i \cdot 2m\pi/n)$ (m and n are incommensurable). When the orbit travels n times, the eigenvalues of the n-time orbit are of the form (1,1,1,1), and then bifurcation occurs. This form of bifurcation phenomenon is especially common in the periodic orbits around the equilateral equilibrium point L_4. Since the trace of the monodromy matrix is $2 + \lambda + 1/\lambda$, it is convenient to use $\mathrm{trace} - 2$ as the stability parameter of the periodic orbit. If the periodic orbit is an n-bifurcation orbit, then $\mathrm{trace} - 2$ will equal to $2\cos(2m\pi/n)$.

In this work, we followed the evolution of the long period family for a chosen set of different mass ratios from μ_5(=0.00550920...) to μ_4(=0.00827037...). If the termination is on a short period family travelling four times, then this is the critical value we expect to find (it is around 0.00814172). Then we chose mass ratios around $\overline{\mu}$ to check the details of the transition of bridges.

3 RESULTS

We found the critical mass ratio value $\overline{\mu}$ to be around 0.00814172. We chose three values around it: 0.008, 0.00812 and 0.0082. For each value, we computed the 4-bifurcation short periodic orbit and computed the bridges connecting the long and short period families. The break-up and recombination of bridges were similar to those found by Henrard (1970), with only slight differences.

For the mass ratio 0.008, the double-lane bridge $B(4S, 4S')$ still exists and the long period family ends on a short period orbit travelling five times (i.e. the bridge $B(L_4, 5S)$ exists). In the left panel of Figure 1, we show the variation of period with the Jacobi constant. The dashed lines indicate the two lanes of bridge $B(4S, 4S')$. For the short period family, we multiply the period four times in the figure. In the right panel we show the variation of the stability parameter with the Jacobi constant. The dashed line indicates one lane of the bridge $B(4S, 4S')$ which is the upper lane shown in the left panel, while the black line indicates the long period family. We can see from the figure that the long period family and the lane do not meet. In the right panel, **A** denotes the periodic orbit in the lane with stability parameter 2, and **B** indicates the hump in the stability curve of the long family. In the following figures, we will see how the point **B** rises to merge with the point **A** to form a new bridge between L_4 and $4S$.

For the mass ratio 0.00812 which is closer to, but still smaller than $\overline{\mu}$, we did the same task as with the mass ratio 0.008. The results are shown in Figure 2. Again, the two lanes of the bridge $B(4S, 4S')$ exist different from Figure 1, the long period family and the lane do intersect here. This is different from the figures in Henrard (1970). The long period family grows sharp humps and does not intersect the lane until the mass ratio reaches the critical one. Comparing the right panels of Figures 1 and 2, we find that hump **B** in the parameter curve of the long family is raised and become closer to the point **A**. It seems that when μ approaches $\overline{\mu}$, the stability parameter of the point **B** approaches 2. In order to show this tendency, we plot the stability parameter of the long period family for mass ratio 0.0081417, which is just short of $\overline{\mu}$. The stability parameter of the hump **B** can reach 1.99816, as shown in Figure 3.

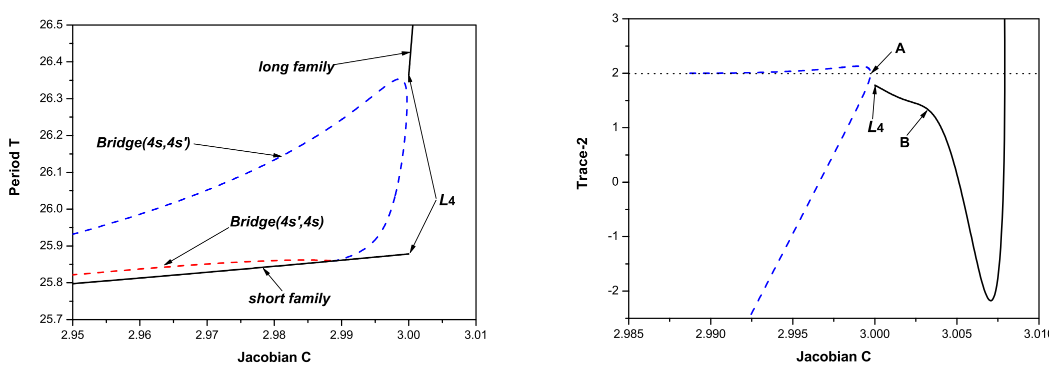

Fig. 1 Periodic families for the mass ratio 0.008.

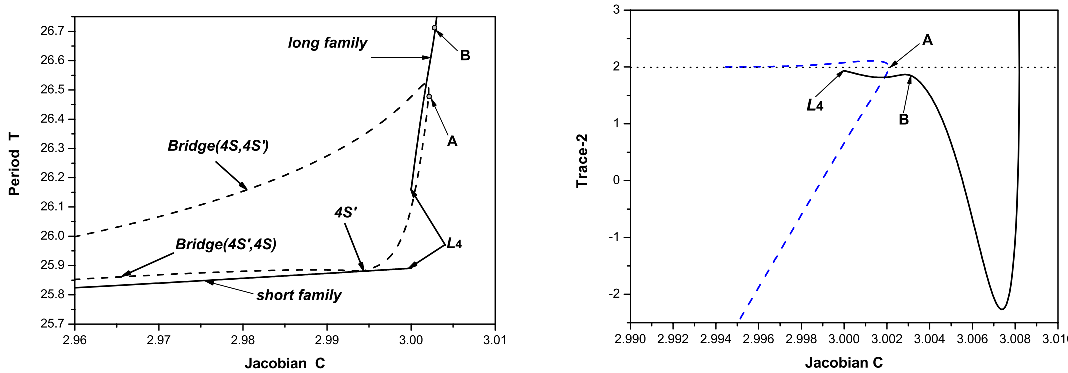

Fig. 2 Periodic families for the mass ratio 0.00812.

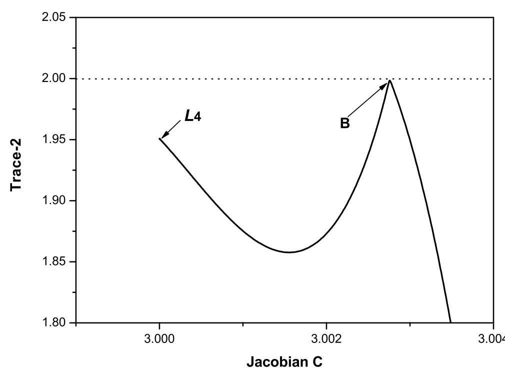

Fig. 3 Stability parameter for the mass ratio 0.0081417.

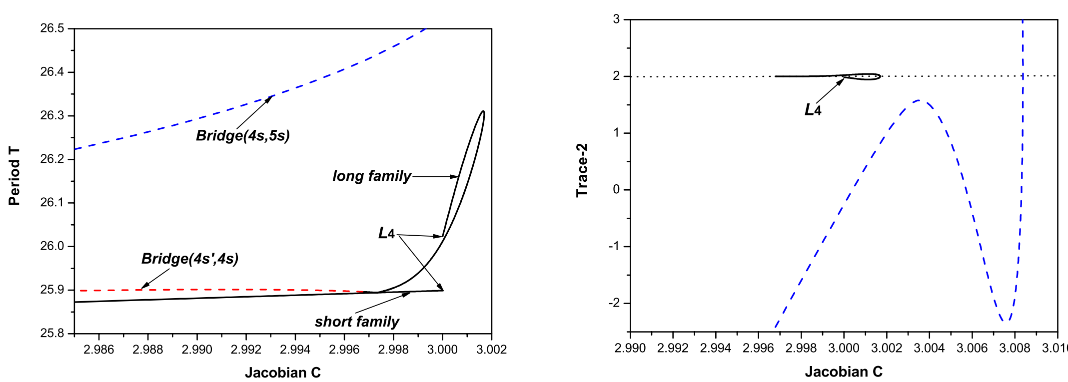

Fig. 4 Periodic families for the mass ration 0.0082.

From these results, it is reasonable to speculate that when μ reaches $\overline{\mu}$, the stability parameter of the hump **B** reaches 2 and it becomes a bifurcation point. It merges with the point **A** to indicate a singular periodic orbit. This singular periodic orbit belongs to four branches of periodic orbits which terminate on L_4, $4S$, $4S'$ and $5S$, respectively.

When μ is larger than $\overline{\mu}$ (μ=0.0082, say), the singular periodic orbit disappears. One lane of the original bridge $B(4S, 4S')$ is broken. One part of the broken bridge merges with part of the original long period family to form the new long period family. The other part of the broken lane merges with the other part of the original long family to form a new bridge $B(4S, 5S)$, but the other lane of the original bridge $B(4S, 4S')$ still exists, as shown in the left panel of Figure 4. The right panel shows a similar phenomenon. The dashed line is the stability parameter curve of the new bridge $B(4S, 5S)$, and the black line is the one for the new long period family. Comparing the right panels of Figures 1, 2 and 4, we can see that the stability curves start to get close, merge, recombine and separate again. In this process points **A** and **B** play crucial roles.

4 CONCLUSIONS

We found a critical value $\overline{\mu}$ between μ_4 and μ_5, which enriches the genealogy of long period family for mass ratios smaller than μ_1 (Henrard (1970)). We gave numerical examples to show how the breakup and recombination of bridges happen for mass ratios around $\overline{\mu}$. We tried to find other critical values between μ_{k+1} and μ_k for $k \geq 5$ without success. It seems that the existence of these critical values is correlated with the hump in the stability curve of the long period family, and $\overline{\mu}$ is the last critical value. We will discuss this later more fully in another paper.

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