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Third Order Effect of Rotation on Stellar Oscillations of a B Star

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Abstract We aim at investigating the effect of rotation up to the third order in the angular velocity of a star on the p and g modes, based on the formalism developed by Soufi et al. Our ultimate goal is the study of oscillations of β Cephei stars which are often rapidly rotating stars. Our results show that the third-order perturbation formalism presented by Soufi et al. should be corrected for some missing terms and some misprints in the equations. As a first step in our study of β Cephei stars, we quantify by numerical calculations the effect of rotation on the oscillation frequencies of a uniformly rotating zero-age main-sequence star with 12 $M_{\odot}$. For an equatorial velocity of $100\ \mathrm{km\,s^{-1}}$, it is found that the second- and third-order corrections for $(l, m) = (2, 2)$, for instance, are of the order of 0.01% of the frequency for radial order $n = 6$ and reaches up to 0.5% for $n = 14$.

Key words: stars: β Cephei variables — stars: oscillation — stars: rotation

1 INTRODUCTION

Pulsating stars on the upper main sequence, particularly δ Scuti and β Cephei stars, are rapid rotators as well as being multimode pulsators. The ratio $\epsilon = \Omega/\omega$ of the rotation rate Ω to the typical frequency of oscillations ω seen in these stars is no longer a small quantity as it is for e.g. the Sun. These stars have typically equatorial velocities $\sim 100\ \mathrm{km\,s^{-1}}$, and oscillation periods from half to a few hours which implies $\epsilon \sim 0.1$, whereas for the Sun $\epsilon \sim 10^{-4}$. The effect of rotation on stellar structure and stellar oscillations is usually calculated through a perturbation analysis in which ϵ is the small parameter. For the Sun a first-order perturbation analysis is sufficient, given the accuracy of observed oscillation frequencies, but for stars it is not. In order to achieve the full potential of astroseismology for testing of models for upper main sequence stars a more careful treatment of the effect of rotation on oscillation frequencies is required.

Rotation not only modifies the structure of the star but also changes the frequencies of normal modes. It removes mode degeneracy creating multiplets of modes. If the rotational angular velocity, Ω, does not have any latitudinal dependence, and the rotation is sufficiently slow, the multiplets show a Zeeman-like equidistant structure. At faster rotation rates non-negligible quadratic effects in Ω cause the position of the centroid frequency of multiplets to shift with respect to that of a non-rotating model of the same star, see Karami et al. (2003).

Our long term goal is to study the oscillation properties of rapidly rotating β Cephei star. In this paper, we start with the study of rotation up to third order in the angular velocity of the star on p and g normal modes. To do this we use the third-order perturbation formalism according to Soufi et al. (1998), hereafter S98, with the correction of some misprints and missing terms in some of their equations. We carry out numerical calculations for the frequency corrections for a zero-age main-sequence (ZAMS) star model with mass $M = 12\,M_{\odot}$. Section 2 gives a brief overview of previous relevant work on rotational perturbation theory for stellar oscillations. Section 3 shows the equations for the effect of rotation on oscillations up to third order and discusses the zeroth-order eigensystem. Section 4 presents corrections to eigenfrequencies and coupling coefficients. The numerical results are presented in Section 5. Section 6 is devoted to concluding remarks. A summary of the differences between the present formulation and that of S98 is given in Appendix A. Lengthy formulae are collected in Appendix B. The hermiticity properties of an oscillating rotating system are discussed in Appendix C.

2 PREVIOUS ROTATIONAL PERTURBATION ANALYSIS

Simon (1969) and Saio (1981) studied the frequency corrections due to rotation up to second order for polytropes. Chlebowski (1978) calculated the same corrections for white dwarf models. Gough & Thompson (1990) studied the effects of rotation and magnetic field on stellar oscillations up to second order. They investigated the linearized adiabatic oscillation equation in a rotating frame under the Cowling approximation and without viscous and resistive dissipative forces. However, they considered only an axisymmetric magnetic field, but allowed for its axis not to coincide with the rotation axis. They concluded that rotation and a magnetic field not only split the degenerate frequency multiplets but also shift the central frequency of each multiplet. The shift arises both from the direct effect of the perturbed inertial and Lorentz forces on the waves, and also because the unperturbed centrifugal and Lorentz forces change the structure of the star to one which is no longer spherically symmetric. Gough & Thompson (1990) succeeded in formulating the differential equations governing the oscillatory motions in a form that is Hermitian. The hermiticity of the oscillation equations for the rotating star was preserved by means of an appropriate mapping of each point in the distorted model to a corresponding point in the spherically symmetric stellar model. Although their main attention was on the effect of a magnetic field on the normal modes, they also performed some numerical calculations of the effect of rotation for three different, latitudinally independent, angular velocity profiles $\Omega(r)$. They found that the effect of the second-order centrifugal distortion changes little with spherical harmonic degree, l, and can be approximated well by the asymptotic estimate. Also, for low l the second-order correction due to the advection term is negligible compared with the second-order centrifugal distortion and only for $l \geq 50$ are the two comparable in magnitude.

Dziembowski & Goode (1992) derived a formalism for calculating the effect of differential rotation on normal modes of rotating stars up to second order. They considered angular-velocity profiles with both radial and latitudinal dependency and found that at faster rotation non-negligible quadratic effects in Ω cause a departure from equidistant splitting. They also obtained generalized asymptotic formulae for g-mode splitting for which the Coriolis term is included in the zero-order treatment. These asymptotic results are relevant for white dwarfs and δ Scuti stars. Dziembowski & Goode (1992) concluded that for solar oscillations the second-order effects are dominated by distortion for $l < 500$.

Soufi et al. (1998) extended the formalism of Dziembowski & Goode (1992) up to third order for a rotation profile that is a function of radius only. Their analysis has two advantages compared with previous investigations. By taking into account parts of the effects of the Coriolis force in the zero-order system, the eigenvalue problem for stellar oscillations can be solved up to cubic order without having to solve successive equations for the eigenfunctions at each order. Also the usual m-degeneracy occurring in the absence of rotation is removed at the lowest order. S98 found that near-degenerate coupling due to rotation only occurs between modes with either the same degree l (and different radial orders) or with modes which differ in degree by 2. The first case involves modes in avoided crossings. The second case concerns modes that have close enough frequencies in the non-rotating model to be shifted into resonance if the rotation is sufficiently rapid. In general they showed that the total coupling comes from three distinct contributions: the Coriolis contribution, the non-spherically-symmetric distortion, and a coupling term which involves a combination of these two effects.

Sobouti (1980) studied the normal modes of rotating fluids up to $O(\Omega^2)$. He argued that the p modes allow a perturbation expansion in Ω, whereas this is not the case for the g modes. From a mathematical point of view, the condition for a perturbation series to converge is that the perturbing operator remains smaller than the unperturbed operator throughout the Hilbert space spanned by the normal modes; this condition is not meet by the g modes. Sobouti & Rezania (2001) considered the toroidal modes of rotating fluids, and showed that: a) At $O(\Omega)$ the neutral toroidal motions of the non-rotating fluid organize themselves into a sequence of modes with a definite (l, m) symmetry, but their radial degeneracy persists at this order. b) Coupling of a given toroidal mode of (l, m) symmetry with $(l \pm 2, m)$ toroidal and with $(l \pm 1, m)$ poloidal modes, as well as the removal of the radial degeneracy come about at $O(\Omega^2)$.

Results of calculation of frequency corrections up to third order were presented for models of δ-Scuti stars by Goupil et al. (2001), Goupil & Talon (2002), Pamyatnykh (2003), and Goupil et al. (2004). Daszyńska-Daszkiewicz et al. (2002) studied the effects of mode coupling due to rotation on photometric parameters (amplitude and phase) of stellar pulsations. They reconfirmed the conclusion of S98 that the most important effect of rotation is coupling between close frequency modes of spherical harmonic degree, l, differing by 2 and of the same azimuthal order, m. They presented some numerical results for a sequence of β Cephei star models with uniform rotation and for two- and three-mode couplings. Their calculations were carried out to cubic order in the ratio of rotation to pulsation frequency, according to the third-order formalism of S98. They concluded that due to the increasing effect of centrifugal distortion with the mode frequency, the coupling between acoustic modes is stronger than between gravity modes.

Reese et al. (2006) checked the effects of rotation due to both the Coriolis and centrifugal accelerations on pulsations of rapidly rotating stars by a non-perturbative method. They showed that the main differences between complete and perturbative calculations come essentially from the centrifugal distortion. Suárez et al. (2006) obtained the oscillation frequencies including corrections for rotation up to the second order in the rotation rate for δ Scuti star models.

3 THIRD ORDER PERTURBATION FORMALISM

Following Unno et al. (1989), the equilibrium state of a rotating star can be characterized by a velocity field

$$\boldsymbol{v}_0 = \boldsymbol{\Omega} \times \boldsymbol{r} = \Omega r \sin\theta \boldsymbol{e}_\varphi \ , \tag{1}$$

where Ω denotes the angular velocity. The rotation axis of the star lies along the $\theta = 0$ axis of a spherical coordinate system (r, θ, φ). Ω is assumed to be independent of latitude and can be written as

$$\boldsymbol{\Omega} = \Omega(r)\boldsymbol{e}_z = \bar{\Omega}[1 + \eta(r)]\boldsymbol{e}_z \ , \tag{2}$$

where $\bar{\Omega}$ is the mean rotation rate. The stationary equation of motion in an inertial frame of reference is

$$-(\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{v}_0 = \frac{\nabla p}{\rho} + \nabla\phi \ , \tag{3}$$

where p, ρ and ϕ are the pressure, density and gravitational potential, respectively. One can show that the left hand side of Equation (3) is equal to the centrifugal acceleration, $\boldsymbol{F}$, in a corotating frame, as

$$\boldsymbol{F} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = r\Omega^2 \sin\theta \boldsymbol{e}_s = -(\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{v}_0 \ , \tag{4}$$

where $\boldsymbol{e}_z$, and $\boldsymbol{e}_s = \sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta$ are unit vectors in the cylindrical coordinates (s, φ, z), see Tassoul (2000).

3.1 Equilibrium Structure of Rotating Stars

The stationary equation of motion, Equation (3), is solved following Chandrasekhar (1933) and Chandrasekhar & Lebovitz (1962) by expanding the equilibrium quantities in terms of Legendre polynomials as

$$f(r, \theta) = \tilde{f}(r) + \epsilon^2 f_2 = \tilde{f}(r) + \epsilon^2 f_{22}(r) P_2(\cos\theta) \ , \tag{5}$$

where $P_2(\cos\theta) = 3/2\cos^2\theta - 1/2$ is the second Legendre polynomial and f can be p, ρ or ϕ. For a rotation rate that is a function of r only, higher-order multipole moments (i.e., $P_4, P_6, ..$) need not be considered.

3.1.1 Spherically Symmetric Distortion

The spherically symmetric part of the equilibrium structure can be obtained by substituting Equation (5) in the θ-independent part of the radial component of Equation (3). The result is

$$\frac{d\tilde{p}}{dr} = -\tilde{\rho} g_{\rm e} \ , \tag{6}$$

where the effective gravity is $g_{\rm e}$,

$$g_{\rm e} = \tilde{g} - \frac{2}{3} r\Omega^2 \ , \tag{7}$$

with

$$\tilde{g} = \frac{d\tilde{\phi}}{dr} = \frac{GM_r}{r^2} \ , \tag{8}$$

and M_r is the mass within radius r. Equation (6) is similar to the equation of hydrostatic equilibrium of a non-rotating star but with the inclusion of the spherically symmetric part of the centrifugal force. Note that other equations governing the quantities of internal structure do not change; hence, as in S98, following the approach described by Kippenhahn & Weigert (1994) a standard evolutionary code only modified according to Equation (6) can be used to follow the evolution. Of course, in order to compute a model at a given age, the profile of the rotation rate is needed which requires the knowledge of its temporal evolution.

3.2 Non-spherically Symmetric Distortion

Substituting Equation (5) in the θ-dependent parts of the radial and the tangential components of Equation (3) and neglecting $O(\epsilon^4)$ terms yields the non-spherically-symmetric part of the equilibrium structure as:

$$p_{22} = -\tilde{\rho} r^2 (\bar{\Omega}/\epsilon)^2 u_2 \ , \tag{9}$$

$$\rho_{22} = \frac{\tilde{\rho} r (\bar{\Omega}/\epsilon)^2}{\tilde{g}} \left(\frac{d\ln\tilde{\rho}}{d\ln r} u_2 + b_2 \right) \ , \tag{10}$$

where

$$\begin{aligned} u_2 &= \frac{\phi_{22}}{r^2}(\bar{\Omega}/\epsilon)^{-2} + \tfrac{1}{3}(1+\eta_2) \ , \\ b_2 &= \tfrac{1}{3} r \frac{d\eta_2}{dr} \ , \end{aligned} \tag{11}$$

with $\eta_2 = \eta(\eta+2)$. The perturbed gravitational potential ϕ_{22} satisfies the perturbed Poisson equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi_{22}}{dr} \right) - \frac{6}{r^2}\phi_{22} = 4\pi G \rho_{22} \ . \tag{12}$$

Equation (12) can be solved by numerical integration and with the appropriate boundary conditions: $\phi_{22} \propto r^2$ at the centre of the star, and $\phi_{22} \propto r^{-3}$ at the surface. Equations (9)–(12) are identical to equations (77)–(80) (with k=1) of Dziembowski & Goode (1992) and also with equations (15)–(17) of S98.

3.3 Zeroth-order Eigensystem

Following S98, we include parts of the Coriolis and non-spherical distortion effects in the zero-order eigensystem. This yields eigenfrequencies ω_0 of eigenmodes which are no longer m-degenerate, even at zero order. The way of building the zero-order eigensystem and the associated basis of eigenmodes enables one to solve the eigenvalue problem up to cubic order without having to solve the successive equations for the eigenfunctions at each order.

A zero-order mode ξ_0 is defined by

$$\xi_0 = \xi_{p0} + \epsilon \xi_{\rm t1} \ , \tag{13}$$

where the poloidal ξ_{p0} and toroidal $\xi_{\rm t1}$ eigenfunctions are characterized by a single spherical harmonic, as in the case of a non-rotating model, and are given by

$$\begin{aligned} \xi_{p0} &= r(yY_l^m + z\nabla_{\rm H} Y_l^m), \\ \xi_{\rm t1} &= r\frac{\bar{\Omega}}{\hat{\omega}_0}(\tau_{l+1} \boldsymbol{e}_r \times \nabla_H Y_{l+1}^m + \hat{\tau}_{l-1} \boldsymbol{e}_r \times \nabla_H Y_{l-1}^m) \ , \end{aligned} \tag{14}$$

where $\nabla_{\rm H} \equiv (\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi)$ is the horizontal part of the gradient operator and $\hat{\omega}_0 = \omega_0 + m\Omega$.

According to equation (39) in S98 the eigenfrequency ω_0 can be rewritten as:

$$\omega_0 = \omega_0^{(0)} + \omega_1 \ , \tag{15}$$

where $\omega_0^{(0)}$ is the usual normal-mode frequency of the system when rotation is absent, and

$$\omega_1 = m\bar{\Omega}(C_{\rm L} - 1 - J_1) \ . \tag{16}$$

The quantities $C_{\rm L}$ and J_1 are given by equation (40) in S98 and references therein. Note that, following S98, we assume a dependence of the perturbations on time t and longitude φ as $\propto \exp[i(m\varphi + \omega t)]$, such that prograde modes have $m < 0$.

The frequency ω_0 includes the first-order correction due to rotation, ω_1, as well as the second-order correction due to spherically symmetric distortion. Also note that ω_1 is the usual first-order frequency shift due to the Coriolis force and can be rewritten as:

$$\omega_1 = -\frac{m}{I}\int \Omega(r)\left[y^2 + (\Lambda - 1)z^2 - 2yz\right] r^4 \tilde{\rho} dr \ , \tag{17}$$

where I is the mode inertia,

$$I = \langle \xi_{p0} \mid \tilde{\rho}\xi_{p0} \rangle = \int dr \tilde{\rho} r^4 (y^2 + \Lambda z^2) \ , \tag{18}$$

with $\Lambda = l(l+1)$.

3.4 Poloidal Eigenfunctions

Following Unno et al. (1989), we define the dimensionless variables y_t, v and w as

$$y_t \equiv \frac{1}{g_{\rm e} r}\left(\tilde{\phi}' + \frac{\tilde{p}'}{\tilde{\rho}}\right) \ , \quad v \equiv \frac{\tilde{\phi}'}{g_{\rm e} r} \ , \quad w \equiv \frac{1}{g_{\rm e}}\frac{d\tilde{\phi}'}{dr} \ . \tag{19}$$

Then as in S98, the expression for the poloidal components is obtained as

$$\begin{aligned} r\frac{dy}{dr} &= (V_g - 3 + h_1)y + (\zeta - V_g)y_t + V_g v \\ &= \lambda - 3y + \Lambda z \ , \end{aligned} \tag{20}$$

$$r\frac{dy_t}{dr} = \left(C_r\hat{\sigma}^2 - A - \frac{h_1^2}{\zeta}\right)y + (A + 1 - U - \chi - h_1)y_t - Av \ , \tag{21}$$

$$r\frac{dv}{dr} = (1 - U - \chi)v + w \ , \tag{22}$$

$$r\frac{dw}{dr} = \frac{UA}{1-\sigma_r}y + \frac{UV_g}{1-\sigma_r}y_t + \left(\Lambda - \frac{UV_g}{1-\sigma_r}\right)v - (U + \chi)w \ , \tag{23}$$

$$\Lambda z - \zeta y_t - h_1 y = 0 \ . \tag{24}$$

Here $\sigma^2 = R^3\omega^2/GM$ is the square of the dimensionless oscillation frequency and

$$\hat{\sigma} \equiv \sigma + m\sigma_\Omega \ , \quad \sigma_\Omega \equiv \frac{\Omega}{\sqrt{GM/R^3}} \ , \tag{25}$$

$$A \equiv \frac{1}{\Gamma_1}\frac{d\ln\tilde{p}}{d\ln r} - \frac{d\ln\tilde{\rho}}{d\ln r} \ , \quad V_g \equiv -\frac{1}{\Gamma_1}\frac{d\ln\tilde{p}}{d\ln r} \ , \tag{26}$$

$$U \equiv \frac{d\ln M_r}{d\ln r} \ , \quad \sigma_r \equiv \frac{2}{3}r\Omega^2/\tilde{g} \ , \tag{27}$$

$$C \equiv \frac{(r/R)^3}{(M_r/M)} \ , \quad C_r \equiv \frac{C}{1-\sigma_r} \ , \tag{28}$$

$$\lambda \equiv V_g(y - y_t + v) \ , \quad \alpha \equiv 2m\frac{\sigma_\Omega}{\hat{\sigma}} \ , \tag{29}$$

$$\chi \equiv \frac{2r\Omega^2}{3g_{\rm e}}\left(U - 3 - \frac{d\ln\Omega^2}{d\ln r}\right) \ , \tag{30}$$

$$\zeta \equiv \frac{\Lambda}{\Lambda-\alpha}\frac{\Lambda}{C_r\hat{\sigma}^2} \ , \quad h_1 \equiv \frac{\Lambda\alpha}{\Lambda-\alpha} \ . \tag{31}$$

Here M and R are the star's mass and radius respectively and G is the gravitational constant.

For radial oscillations ($l = \Lambda = 0$) the Poisson equation for ξ_{p0}, Equation (12), reduces to

$$\frac{d\tilde{\phi}'}{dr} = -\tilde{g}Uy \ . \tag{32}$$

Thus $\tilde{\phi}'$ can be eliminated from the problem; instead of Equation (19) we define the dimensionless variable y_t in this case as

$$y_t = \frac{\tilde{p}'}{g_{\rm e}r\tilde{\rho}} \ , \tag{33}$$

obtaining

$$r\frac{dy}{dr} = (V_g - 3)y - V_g y_t \ , \tag{34}$$

$$r\frac{dy_t}{dr} = \left(C_r\hat{\sigma}^2 - A + \frac{U}{1-\sigma_r}\right)y + (A + 1 - U - \chi)y_t \ . \tag{35}$$

For radial oscillations there is no transverse component, i.e., $z = 0$.

3.5 Toroidal Eigenfunctions

Following again S98 the components τ, $\hat{\tau}$ of the toroidal part, ξ_{t1}, for a mode $k \equiv (n_k, l_k, m_k)$, are obtained as follows:

$$\begin{aligned}\tau_{k+1} &\equiv \tau_{n_k,l_k+1,m_k} \\ &= i\frac{\beta_{k+1}}{(\Lambda_{l_k+1} - \alpha_k)}\left(2P_k + 3m_k\frac{\bar{\Omega}}{\hat{\omega}_0}d_k\right) \ ,\end{aligned} \tag{36}$$

$$\begin{aligned}\hat{\tau}_{k-1} &= \hat{\tau}_{n_k,l_k-1,m_k} \\ &= i\frac{\beta_k}{(\Lambda_{l_k-1} - \alpha_k)}\left(2\hat{P}_k + 3m_k\frac{\bar{\Omega}}{\hat{\omega}_0}d_k\right) \ ,\end{aligned} \tag{37}$$

with

$$P_k = (1+\eta)(l_k+2)(-y_k + l_k z_k) \ , \tag{38}$$

$$\hat{P}_k = (1+\eta)(l_k-1)(y_k + (l_k+1)z_k) \ , \tag{39}$$

$$\beta_k = \sqrt{\frac{(l_k^2 - m_k^2)}{4l_k^2 - 1}} \ , \tag{40}$$

β_{k+1} being defined similarly, but with l_k replaced by $l_k + 1$, and

$$d_k = \left(\frac{g_{\rm e}}{\tilde{g}}v_k + y_k - C\hat{\sigma}_k^2 z_k\right)\left(\frac{d\ln\tilde{\rho}}{d\ln r}u_2 + b_2\right) + \lambda_k u_2 \ . \tag{41}$$

4 CORRECTION TO EIGENFREQUENCIES AND COUPLING COEFFICIENTS

As discussed in S98, in general the correction to the eigenfrequencies must take into account coupling between nearby modes. Thus we need to evaluate the coupling coefficients between different modes. However, in the case of an isolated mode, k, the correction $\omega_{\rm c}$, to the eigenfrequency $\omega = \omega_0 + \omega_{\rm c}$, is obtained from

$$\begin{aligned} \omega_{\rm c} &= \frac{H_{kk}}{2I\omega_0^{(0)}} \equiv \mathcal{H}_{kk} \\ &= \omega^{\rm T} + \omega^{\rm D} + \omega^{\rm C} + O(\epsilon^4) \ , \end{aligned} \tag{42}$$

where $\mathcal{H}_{kq}$ is the matrix of the coupling coefficients and is defined as

$$\mathcal{H}_{kq} \equiv \frac{H_{kq}}{2J_{kq}} = \mathcal{T}_{kq} + \mathcal{D}_{kq} + \mathcal{C}_{kq} + O(\epsilon^4) \ , \tag{43}$$

and $J_{kq} = \sqrt{\omega_{0k}^{(0)}\omega_{0q}^{(0)} I_k I_q}$. Note that for an isolated mode each contribution $\omega^{\rm T} = \mathcal{T}_{kk}$, $\omega^{\rm D} = \mathcal{D}_{kk}$ and $\omega^{\rm C} = \mathcal{C}_{kk}$ to the frequency correction arises from the corresponding diagonal term of the interaction matrix $\mathcal{H}_{kk}$.

In order to obtain expressions for the different contributions to the total coupling coefficients $\mathcal{H}_{kq}$, it is convenient (see equations (B1)–(B2) in S98), to define a dimensionless radius x and density $\bar{\tilde{\rho}}$:

$$\begin{aligned} x &\equiv r/R \ , \\ \bar{\tilde{\rho}} &\equiv \tilde{\rho}/\rho_{\rm c} \ , \end{aligned} \tag{44}$$

where $\rho_{\rm c}$ is the value of the density at the centre of the (deformed) star in equilibrium. The second-order perturbation terms of pressure, density and gravitational potential are also made dimensionless as follows:

$$\begin{aligned} \bar{p}_{22} &\equiv p_{22}/p_{\rm c} \ , \\ \bar{\rho}_{22} &\equiv \rho_{22}/\rho_{\rm c} \ , \\ \bar{\phi}_{22} &\equiv \phi_{22}/\phi_{\rm c} \ , \end{aligned} \tag{45}$$

where $p_{\rm c} = \rho_{\rm c}\phi_{\rm c}$ and $\phi_{\rm c} = R^2\bar{\Omega}^2$. The oscillation frequencies are made dimensionless with the dynamical time scale of the star:

$$\begin{aligned} \sigma_0^{(0)} &\equiv \frac{\omega_0^{(0)}}{\sqrt{GM/R^3}} \ , \\ \sigma_{\bar{\Omega}} &\equiv \frac{\bar{\Omega}}{\sqrt{GM/R^3}} \ . \end{aligned} \tag{46}$$

The dimensionless mode inertia terms are:

$$\begin{aligned} \bar{I}_k &\equiv \frac{I_k}{\rho_{\rm c}R^5} \ , \\ \bar{J}_{kq} &\equiv \frac{J_{kq}}{\rho_{\rm c}R^5\sqrt{GM/R^3}} \\ &= \sqrt{\sigma_{0k}^{(0)}\sigma_{0q}^{(0)}\bar{I}_k\bar{I}_q} \ . \end{aligned} \tag{47}$$

4.1 Coriolis Contribution: $\mathcal{T}_{kq}$

As in S98 the elements $\mathcal{T}_{kq}$ are obtained as

$$\begin{aligned} \overline{\mathcal{T}}_{kq} &\equiv \frac{\mathcal{T}_{kq}}{\sqrt{GM/R^3}} \\ &= \overline{\mathcal{T}}_{qk} = \delta_{l_k l_q}\overline{T}^{(1)} + \delta_{l_k l_q+2}\overline{T}^{(2)}_{kq} + \delta_{l_k l_q-2}\overline{T}^{(2)*}_{qk} \ , \end{aligned} \tag{48}$$

where $*$ denotes complex conjugate; here the diagonal and off-diagonal terms are:

$$\begin{aligned}\overline{T}^{(1)} =\ & \frac{\sigma_{\bar{\Omega}}^2}{2J_{kq}}\left\{\int_0^1 dx\bar{\bar{\rho}}x^4\times\right.\\ & \quad\times(\Lambda_{k+1}\tau_{k+1}^*\tau_{q+1}+\Lambda_{k-1}\hat{\tau}_{k-1}^*\hat{\tau}_{q-1})\\ & -\frac{4m\sigma_{\bar{\Omega}}}{\sigma_{0k}+\sigma_{0q}}\int_0^1 dx\bar{\bar{\rho}}x^4(1+\eta)\times\\ & \quad\left.\times(\tau_{k+1}^*\tau_{q+1}+\hat{\tau}_{k-1}^*\hat{\tau}_{q-1})\right\} ,\end{aligned} \tag{49}$$

$$\begin{aligned}\overline{T}_{kq}^{(2)} =\ & \frac{\sigma_{\bar{\Omega}}^2}{2J_{kq}}\left\{\int_0^1 dx\bar{\bar{\rho}}x^4\Lambda_{k-1}\hat{\tau}_{k-1}^*\tau_{q+1}\right.\\ & \left.-\frac{4m\sigma_{\bar{\Omega}}}{\sigma_{0k}+\sigma_{0q}}\int_0^1 dx\bar{\bar{\rho}}x^4(1+\eta)\hat{\tau}_{k-1}^*\tau_{q+1}\right\} .\end{aligned} \tag{50}$$

We recall that some effects of distortion are already included in the toroidal zeroth-order system. Therefore, distortion contributes indirectly through the toroidal part of the eigenfunctions, i.e., through the components τ and $\hat{\tau}$.

Note that the case $k \neq q$ represents a coupling between near-degenerate modes. Equation (48) shows that the coupling occurs between modes either with same degree l (and generally different radial orders) or with degrees that differ by ± 2. The frequency correction, ω^{T}, for a single mode can be obtained from the diagonal element $\mathcal{T}_{kk}$ as

$$\sigma^{\mathrm{T}} = \frac{\omega^{\mathrm{T}}}{\sqrt{GM/R^3}} = \overline{\mathcal{T}}_{kk} \equiv \sigma_2^{\mathrm{T}} + \sigma_3^{\mathrm{T}} , \tag{51}$$

with the second-order contribution,

$$\begin{aligned}\sigma_2^{\mathrm{T}} = \tfrac{1}{2}(\frac{\sigma_0}{I})(\frac{\sigma_{\bar{\Omega}}}{\sigma_0})^2\int_0^1 dx\bar{\bar{\rho}}x^4\ & (\Lambda_{k+1}\mid\tau_{k+1}\mid^2\\ & +\Lambda_{k-1}\mid\hat{\tau}_{k-1}\mid^2) ,\end{aligned} \tag{52}$$

and the third-order contribution

$$\begin{aligned}\sigma_3^{\mathrm{T}} =\ & m(\frac{\sigma_0}{I})(\frac{\sigma_{\bar{\Omega}}}{\sigma_0})^3\left\{\frac{C_{\mathrm{L}}-1-J_1}{2}\int_0^1 dx\bar{\bar{\rho}}x^4\times\right.\\ & \times(\Lambda_{k+1}\mid\tau_{k+1}\mid^2+\Lambda_{k-1}\mid\hat{\tau}_{k-1}\mid^2)\\ & \left.-\int_0^1 dx\bar{\bar{\rho}}x^4(1+\eta)(\mid\tau_{k+1}\mid^2+\mid\hat{\tau}_{k-1}\mid^2)\right\} .\end{aligned} \tag{53}$$

Equations (48)–(53) are identical with equations (B3)–(B7) of S98.

Note that to separate the frequency correction into second and third orders in respect of $\epsilon = \sigma_{\bar{\Omega}}/\sigma_0$, one also needs to use the following approximations:

$$\begin{aligned}\frac{\omega_0}{\omega_0^{(0)}} = \frac{\sigma_0}{\sigma_0^{(0)}} & \simeq 1+\left(1-\frac{\sigma_0^{(0)}}{\sigma_0}\right)\\ & = 1+\frac{\sigma_{\bar{\Omega}}}{\sigma_0}\left(\frac{\sigma_0-\sigma_0^{(0)}}{\sigma_{\bar{\Omega}}}\right)\\ & = 1+m\frac{\bar{\Omega}}{\omega_0}(C_{\mathrm{L}}-1-J_1) ,\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{\sigma_{\bar{\Omega}}^2}{J} & \simeq \left(\frac{\sigma_0}{I}\right)\left(\frac{\sigma_{\bar{\Omega}}}{\sigma_0}\right)^2\left[1+\frac{\sigma_{\bar{\Omega}}}{\sigma_0}\left(\frac{\sigma_0-\sigma_0^{(0)}}{\sigma_{\bar{\Omega}}}\right)\right]\\ & \propto O(\bar{\Omega}^2)+O(\bar{\Omega}^3) ,\end{aligned} \tag{55}$$

in which the following has been used:

$$\frac{\sigma_0-\sigma_0^{(0)}}{\sigma_{\bar{\Omega}}} = m(C_{\mathrm{L}}-1-J_1) \propto O\left(\frac{\bar{\Omega}}{\bar{\Omega}}\right) = O(1) . \tag{56}$$

4.2 Non-spherically Symmetric Distortion: $\mathcal{D}_{kq}$

Following again S98, coefficients $\mathcal{D}_{kq}$ can be obtained in an explicitly symmetric form as follows:

$$\begin{aligned}\mathcal{D}_{kq} = \mathcal{D}_{qk} = \tfrac{1}{2J_{kq}}\int d^3\boldsymbol{x}\Big\{&(\Gamma_1 p_2)\nabla\cdot\xi^*_{p0k}\nabla\cdot\xi_{p0q}+\\ &\rho_2(\xi_{p0q}\cdot\nabla\tilde{\phi}'^*_k+\xi^*_{p0k}\cdot\nabla\tilde{\phi}'_q)\\ &-\frac{1}{\tilde{\rho}}\nabla p_2\cdot(\xi_{p0q}\tilde{\rho}'^*_k+\xi^*_{p0k}\tilde{\rho}'_q)+\tilde{\rho}r^2W\\ &-\rho_2\hat{\omega}_0^2\xi^*_{p0k}\cdot\xi_{p0q}\Big\}\ ,\end{aligned} \tag{57}$$

where

$$\begin{aligned}W = &\ y_k y_q Y^*_k Y_q\left[P_2 w_1+(1-P_2)\frac{2}{3}r\frac{d\Omega^2}{dr}\right]-\frac{1}{3}r\frac{d\Omega^2}{dr}\times\\ &\times\ (y_k z_q Y^*_k\nabla_{\mathrm{H}}Y_q+y_q z_k Y_q\nabla_{\mathrm{H}}Y^*_k)\cdot\nabla_{\mathrm{H}}P_2\ ,\end{aligned} \tag{58}$$

where $Y_k \equiv Y^{m_k}_{l_k}$, P_2 is the second Legendre polynomial and w_1 is defined by

$$w_1 = \frac{d}{dr}\left(\frac{1}{\tilde{\rho}}\right)\frac{dp_{22}}{dr}+\frac{1}{\tilde{\rho}}\frac{d}{dr}\left(\frac{\rho_{22}}{\tilde{\rho}}\right)\frac{d\tilde{p}}{dr}\ . \tag{59}$$

Equations (57)–(59) are identical to equations (B8)–(B10) in S98. Note that $\mathcal{D}_{kq}$ is non-zero only when $m_k = m_q$.

The angular parts of the integral Equation (57) can be evaluated analytically. Finally the result can be written in dimensionless form as follows:

$$\begin{aligned}\overline{\mathcal{D}}_{kq} \equiv &\ \frac{\mathcal{D}_{kq}}{\sqrt{GM/R^3}}\\ = &\ \delta_{l_k l_q}\bar{D}+\delta_{l_k l_q+2}(\tfrac{3}{2}\beta_k\beta_{q+1})\bar{D}_{kq}\\ &+\delta_{l_k l_q-2}(\tfrac{3}{2}\beta_{k+1}\beta_q)\bar{D}_{kq}\ ,\end{aligned} \tag{60}$$

with

$$\bar{D} = \mathcal{Q}_{kk2}\bar{D}_{kq}+\frac{\sigma^2_{\bar{\Omega}}}{\bar{J}_{kq}}\int_0^1 dx\bar{\tilde{\rho}}x^4 y_k y_q b_2\ , \tag{61}$$

and

$$\bar{D}_{kq} = \bar{D}^{(1)}_{kq}+\bar{D}^{(2)}_{kq}+\bar{D}^{(3)}_{kq}+\bar{D}^{(4)}_{kq}+\bar{D}^{(5)}_{kq}\ , \tag{62}$$

where the expressions for the different terms appearing in $\bar{D}_{kq}$ are given in Appendix B.

The diagonal elements $\overline{\mathcal{D}}_{kk}$ reduce to the frequency correction ω^{D} as

$$\sigma^{\mathrm{D}} = \frac{\omega^{\mathrm{D}}}{\sqrt{GM/R^3}} = \overline{\mathcal{D}}_{kk} = \sigma^{\mathrm{D}}_2+\sigma^{\mathrm{D}}_3\ , \tag{63}$$

where the second-order contribution is

$$\sigma^{\mathrm{D}}_2 = \left(\frac{\sigma_0}{\bar{I}}\right)\left(\frac{\sigma_{\bar{\Omega}}}{\sigma_0}\right)^2\left(\frac{\bar{J}_{kk}}{\sigma^2_{\bar{\Omega}}}\right)\bar{D}\ , \tag{64}$$

and the third-order contribution is

$$\begin{aligned}\sigma^{\mathrm{D}}_3 &= \left(\frac{\sigma_0}{\bar{I}}\right)\left(\frac{\sigma_{\bar{\Omega}}}{\sigma_0}\right)^3\left(\frac{\sigma_0-\sigma^{(0)}_0}{\sigma_{\bar{\Omega}}}\right)\left(\frac{\bar{J}_{kk}}{\sigma^2_{\bar{\Omega}}}\right)\bar{D}\\ &= m(C_{\mathrm{L}}-1-J_1)\left(\frac{\sigma_0}{\bar{I}}\right)\left(\frac{\sigma_{\bar{\Omega}}}{\sigma_0}\right)^3\left(\frac{\bar{J}_{kk}}{\sigma^2_{\bar{\Omega}}}\right)\bar{D}\ ,\end{aligned} \tag{65}$$

in which $\bar{D}$ is given by Equation (61).

Note that Equations (60)–(65) differ from equations (B12)–(B21) in S98. The differences come from the fact that S98 carried out further integration by parts to remove $d\ln\tilde{\rho}/d\ln r = -(A+V_g)$ which sometimes can cause numerical oscillations of the integrand of Equation (57). For the models we considered (see Sect. 5) no such numerical problem exists and we choose to compute the coefficient directly from the definition in Equation (57). We furthermore note that the partial integration gives rise to surface terms which are neglected in S98; in fact, for models without an extended atmosphere they become considerable. We return to this subject in Appendix A1.

Also, in the third-order terms of the relations, the approximation $z \simeq y_t/C\sigma_0^2$, which is identical with the non-rotating case, instead of its exact relation Equation (24) was used by S98. We find in our numerical results that this is not true and Equation (24) must be used (see Sect. A2).

4.3 Distortion and Coriolis Coupling: $\mathcal{C}_{kq}$

As in S98, the coupling coefficients $\mathcal{C}_{kq}$ can be obtained as:

$$\begin{aligned}\overline{\mathcal{C}}_{kq} = \overline{\mathcal{C}}_{qk} \equiv& \frac{\mathcal{C}_{kq}}{\sqrt{GM/R^3}} \\ &= \delta_{l_k l_q}\bar{C} + \delta_{l_k l_q+2}(\tfrac{3}{2}\beta_k\beta_{q+1})\bar{C}_{kq} \\ &\quad + \delta_{l_k l_q-2}(\tfrac{3}{2}\beta_{k+1}\beta_q)\bar{C}_{kq}\ ,\end{aligned} \tag{66}$$

where

$$\begin{aligned}\bar{C} =& \frac{m}{2J_{kq}}\sigma_\Omega^3(\sigma_{0k}+\sigma_{0q})\int_0^1 dx\bar{\tilde{\rho}}x^4C(1+\eta)\times \\ &\times[b_2-(A+V_g)u_2]\times \\ &\times[z_kz_q+\mathcal{Q}_{kq2}(y_kz_q+y_qz_k+3z_kz_q)]\ ,\end{aligned} \tag{67}$$

$$\begin{aligned}\bar{C}_{kq} =& \frac{m}{2J_{kq}}\sigma_\Omega^3(\sigma_{0k}+\sigma_{0q})\int_0^1 dx\bar{\tilde{\rho}}x^4C(1+\eta)\times \\ &\times[b_2-(A+V_g)u_2][y_kz_q+y_qz_k+3z_kz_q],\end{aligned} \tag{68}$$

an expression for $\mathcal{Q}_{kq2} \equiv \int \sin\theta d\theta d\phi Y_k^* Y_q P_2$ is given by equation (B18) of S98. The diagonal coefficients correspond to the contribution to the frequency correction $\omega^{\mathrm{C}} = \mathcal{C}_{kk}$. This contribution has only terms of third order,

$$\begin{aligned}\sigma^{\mathrm{C}} &= \overline{\mathcal{C}}_{kk} \equiv \frac{\omega^{\mathrm{C}}}{\sqrt{GM/R^3}} = \sigma_3^{\mathrm{C}} \\ &= \frac{m}{I}\sigma_\Omega^3\int_0^1 dx\bar{\tilde{\rho}}x^4C(1+\eta)[b_2-(A+V_g)u_2]\times \\ &\quad\times[z^2+\mathcal{Q}_{kk2}(2yz+3z^2)]\ .\end{aligned} \tag{69}$$

Equations (66)–(68) are identical to equation (B22) in S98. However, Equation (69) differs from equation (B25) in S98 where again a partial integration had been carried out to eliminate the density derivative.

Note that Equations (53), (65) and (69) show that for the case $m = 0$ there are no third-order contributions to the frequency corrections.

The expressions for the separate terms in $\mathcal{H}_{kq}$ (see Eq. (43)), i.e., Equations (48), (57) and (66), are all explicitly symmetric and real-valued. Therefore, one can conclude that the total coupling coefficient is symmetric, real-valued and therefore also Hermitian, $\mathcal{H}_{kq} = \mathcal{H}_{qk}$.

4.4 Close Frequencies and Mode Coupling

The existence of close frequencies can lead to large values of the perturbation terms to the extent that it invalidates the perturbative approach outlined in the previous sections. For such cases one needs to use a degenerate perturbation formalism. Following S98, the zeroth-order eigenfunction of an individual mode is written as a superposition of degenerate eigenfunctions,

$$\xi = \sum_k \mathcal{A}_k\xi_{0k}\ , \tag{70}$$

where ξ_{0k} is the degenerate eigenfunction of the zeroth-order system with amplitudes, $\mathcal{A}_k$, the expansion coefficients normalizable to 1. As in S98, the solution of the eigenvalue problem for the eigenfunction ξ, corresponding to an eigenvalue ω, satisfies the following linear system:

$$\mathcal{A}_k\mu_{kk} + \sum_{q\neq k}\mathcal{A}_q\mu_{kq} = 0 \ , \tag{71}$$

where the components μ_{kq} are defined by

$$\begin{aligned} \mu_{kk} &\equiv (\omega_k - \omega)2\omega_{0k}^{(0)} I_k \ , \\ \mu_{kq} &\equiv 2J_{kq}\mathcal{H}_{kq} \ , \qquad \text{for } q \neq k \ . \end{aligned} \tag{72}$$

One notes that $\mu_{kq} = \mu_{qk}$, because of the symmetric properties of $\mathcal{H}_{kq}$. The eigenvalues, ω, can be obtained from the existence condition of nontrivial solutions of Equations (71), i.e.,

$$\det[\mu_{ij}] = 0 \ , \qquad i,j = 1,\ldots,N \ , \tag{73}$$

where $\det$ is the determinant and N is the total number of the near-degenerate modes.

Here we consider the coupling between $N = 2$ modes. Equations (73) reduces to

$$(\omega_1 - \omega)(\omega_2 - \omega) - \mathcal{H}_{12}^2 = 0 \ . \tag{74}$$

In this case the eigenfrequencies, $\omega = \omega_{\pm}$, can be obtained trivially from the zeroth-order near-degenerate eigenfrequencies and the total corrections due to rotation and coupling are

$$\omega_{\pm} = \left(\frac{\omega_1 + \omega_2}{2}\right) \pm \sqrt{\left(\frac{\omega_1 - \omega_2}{2}\right)^2 + \mathcal{H}_{12}^2} + O(\epsilon^4) \ . \tag{75}$$

Also, the normalized amplitude $bm\mathcal{A} \equiv (\mathcal{A}_1, \mathcal{A}_2)$ appearing in Equation (70) can be derived from Equation (71) as

$$\begin{aligned} \mathcal{A}_1^{(\pm)} &= [1 + (\mu_{11}^{(\pm)}/\mu_{12})^2]^{-1/2} \\ &= [1 + (\mu_{12}/\mu_{22}^{(\pm)})^2]^{-1/2}, \\ \mathcal{A}_2^{(\pm)} &= -(\mu_{11}^{(\pm)}/\mu_{12})\mathcal{A}_1^{(\pm)} \\ &= -(\mu_{12}/\mu_{22}^{(\pm)})\mathcal{A}_1^{(\pm)} \ , \end{aligned} \tag{76}$$

where the quantities labeled $(\pm)$ are related to the eigenvalues $\omega_{\pm}$.

The coupling coefficients $\mathcal{H}_{kq}$ appearing in Equation (72) show that near-degenerate coupling only occurs between modes with the same m and with either the same degree l (and different radial orders) or with degrees differring by ± 2.

5 OSCILLATIONS OF A RAPIDLY ROTATING B STAR

In order to calculate the effect of rotation on normal modes, I consider a uniformly rotating, 12 $M_{\odot}$, ZAMS model generated by the evolution code of Christensen-Dalsgaard (1982) (see also Christensen-Dalsgaard & Thompson 1999). The parameters of the model are listed at Table 5.

The behaviors of some of the equilibrium quantities of the model with the fractional radius, $x = r/R$ are shown in Figure 1. It shows that: 1) From the centre to radius $x = 0.25$, the star is in a convective regime where $N^2 < 0$ and, outside of this radius it is in a radiative regime where $N^2 > 0$; 2) The spherically symmetric density ρ decreases smoothly from its maximum value to nearly zero at the surface. 3) The non-spherically-symmetric correction to the density ρ_{22} decreases to a negative maximum value at $x = 0.3$ and then slowly increases to zero at $x \simeq 0.9$; 4) The non-spherically-symmetric correction to the gravitational potential ϕ_{22} is obtained as the numerical solution of the Poisson relation, Equation (12), by a Runge-Kutta method with an adaptive step-size control. The absolute value of ϕ_{22} increases smoothly to its maximum value at the surface.

Table 1 Stellar parameters of a rotating zero-age main-sequence star in solar units. M, $M_{\rm conv}$, R, $R_{\rm conv}$, $p_{\rm c}$, and $\rho_{\rm c}$ are the total mass, the mass of convective core, the radius, the radius of convective core, the central pressure and density, and $\odot$ denotes solar values; $\sigma_{\bar{\Omega}}$ and ϵ are the dimensionless mean angular velocity and the perturbational expansion coefficient; $T_{\rm dyn}$, $T_{\rm rot}$, and $V_{\rm rot}$ are the dynamical time scale (free fall time), the equatorial period and velocity, respectively. For comparison note that $T_{\odot{\rm dyn}} = 0.5$ h, $T_{\odot{\rm rot}} = 25$ d, and $V_{\odot{\rm rot}} = 2\ {\rm km\,s^{-1}}$.

$M = 12\ M_\odot$	$M_{\rm conv} = 0.34\ M$
$R = 4.32\ R_\odot$	$R_{\rm conv} = 0.25\ R$
$p_{\rm c} = 3.72 \times 10^{-1}\ p_{\odot{\rm c}}$	$\rho_{\rm c} = 4.83 \times 10^{-2}\ \rho_{\odot{\rm c}}$
$\sigma_{\bar{\Omega}} = 1.38 \times 10^{-1}$	$\epsilon = \Omega/\omega = \sigma_{\bar{\Omega}}/2\pi = 2.2 \times 10^{-2}$
$T_{\rm dyn} = \sqrt{R^3/GM} = 1.15$ h	$T_{\rm rot} = 2\pi R/V_{\rm rot} = 2.15$ d
$V_{\rm rot} = R\Omega = R\sigma_{\bar{\Omega}}/T_{\rm dyn} = 100\ {\rm km\,s^{-1}}$	

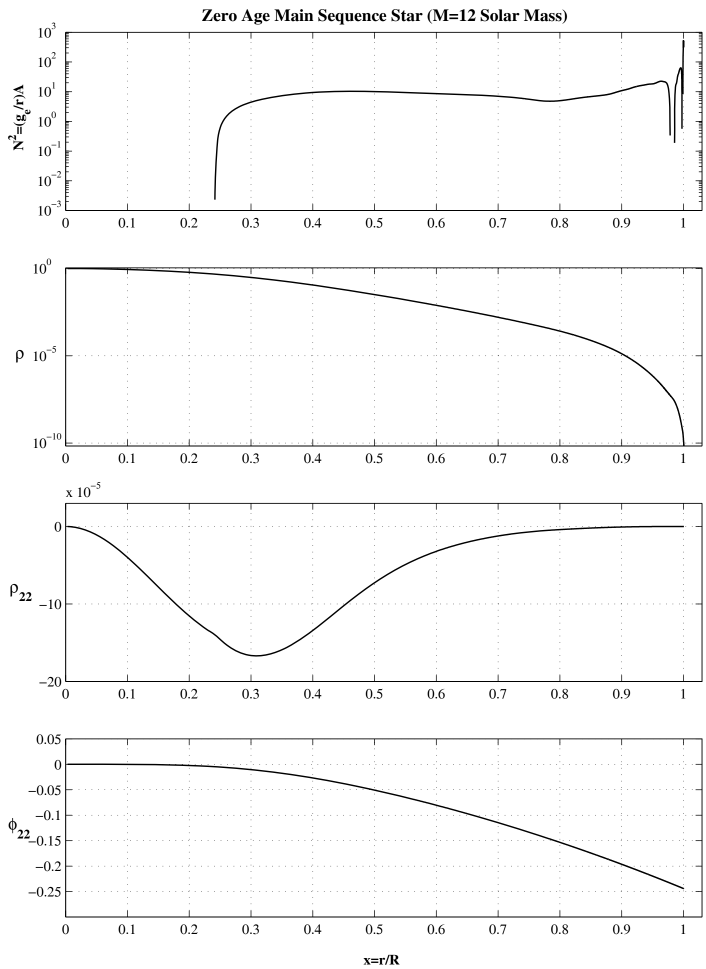

Fig. 1 Dimensionless equilibrium quantities including squared buoyancy frequency $N^2 = (g_{\rm e}/r)A$ with $A = (1/\Gamma_1)(d\ln p/d\ln x) - (d\ln\rho/d\ln x)$, spherically and non-spherically-symmetric contributions ρ and ρ_{22} to the density, and non-spherically-symmetric gravitational-potential contribution ϕ_{22}, in units of GM/R^3, $\rho_{\rm c}$ and $R^2\bar{\Omega}^2$, respectively, against fractional radius $x = r/R$ for a zero-age main-sequence star with $M = 12 M_\odot$.

5.1 Eigenfunctions

The zero-order eigenfunctions are computed from the zero order eigenvalue problem with the pulsation code of Christensen-Dalsgaard (see Christensen-Dalsgaard & Berthomieu 1991), modified according to Equations (20)–(24).

In Figure 2 the radial (y) and horizontal (z) components of the zero-order poloidal eigenfunctions as well as the quantity $r\rho^{1/2}\xi_r/(R^2\rho_c^{1/2})$ related to the radial energy density, where $\xi_r = ry$ is the radial displacement, are plotted against the fractional radius $x = r/R$ for the selected f and p modes with $(l, m) = (2, 2)$ and $n = (0, 1, 8)$. Note that in the case of the radial oscillation ($l = 0$), the results are derived from the reduced set of differential equations composed of Equations (34) and (35).

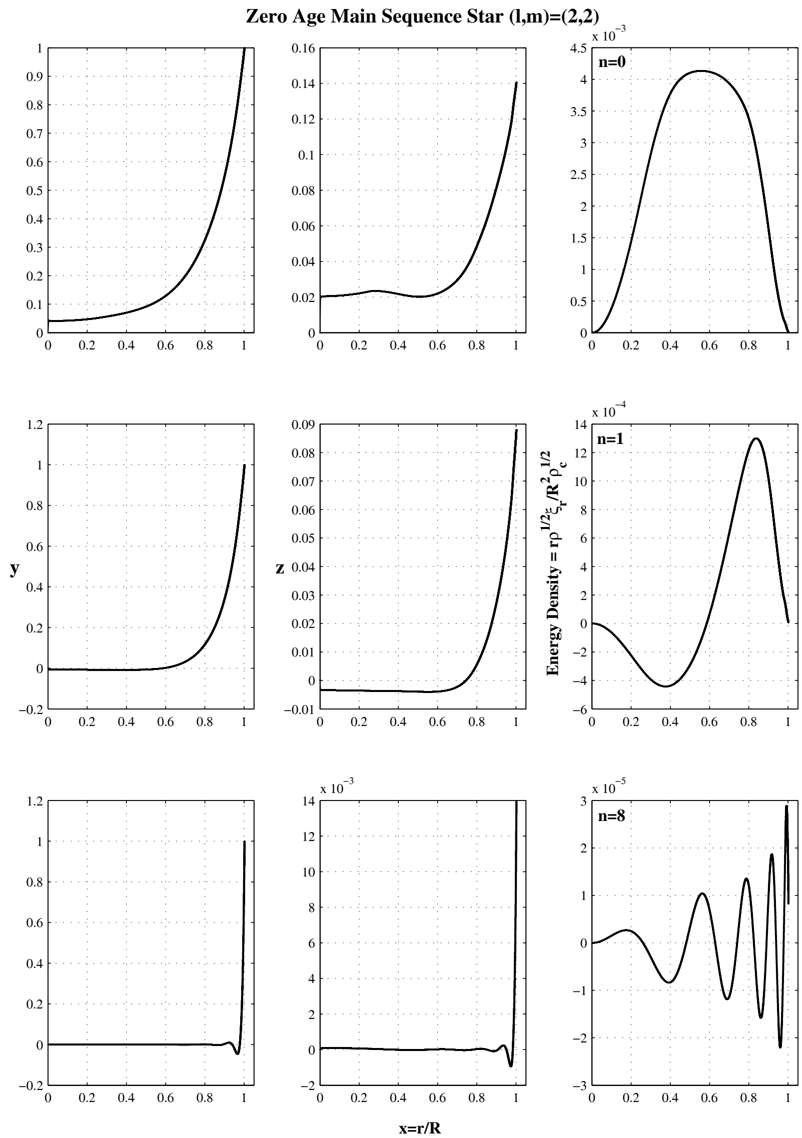

Fig. 2 Zero-order radial y (left) and horizontal z components (middle) and $r\rho^{1/2}\xi_r/(R^2\rho_c^{1/2})$, related to the energy density (right), as functions of the fractional radius $x = r/R$ for selected f and p modes with $(l, m) = (2, 2)$ and $n = (0, 1, 8)$ for the model described in Table 5.

Figure 2 shows that: 1) The number of nodes coincides with the radial order n. 2) At lower radial order, the energy is distributed throughout most of the star. At higher radial order, the displacement is large only in the outer part of the model. 3) The magnitude of the horizontal component z decreases with increasing n (see also Sobouti 1980). In other words, the higher-order p modes oscillate radially rather than horizontally because they are acoustic and longitudinal waves.

5.2 Eigenfrequencies and Corrections

The zero-order eigenfrequency, σ_0, is derived from numerical solutions of Equations (20)–(24) by the modified pulsation code; note that using the eigensystem in Equations (20)–(24) the first-order frequency correction, σ_1, is implicitly included in σ_0 (see Eqs. (15) and (16)). The second- and third-order Coriolis contributions, $(\sigma_2^{\mathrm{T}},\ \sigma_3^{\mathrm{T}})$, the second- and third-order non-spherically-symmetric distortions, $(\sigma_2^{\mathrm{D}},\ \sigma_3^{\mathrm{D}})$, and the third-order distortion and Coriolis coupling, σ_3^{C}, are derived from numerical integrations of Equations (52),(53),(64),(65) and (69).

In Tables 2 to 5.2 the results of different contributions of frequency corrections due to effect of rotation up to third order are tabulated. Included are the selected p modes with $(l, m) = (0, 0)$, $(1, 1)$, $(2, 1)$ and $(2, 2)$ with $n = (0, \dots, 14)$, $(1, \dots, 15)$, and $(0, \dots, 14)$. The fundamental modes are known with $n = 0$ and labelled f. The modes with $n \geq 1$ are labelled by $\mathrm{p}_1, \dots, \mathrm{p}_n$.

Tables 2 to 5.2 show that: 1) The values of zero-order eigenfrequency, σ_0, increase as the radial order n increases. 2) For the case $m = 0$, the odd-order frequency corrections, i.e., in the present case those of first and third order, vanish. In this case only the second-order frequency corrections exist (see Eqs. (17), (52), (53), (64), (65), and (69)). 3) For radial oscillation ($l = 0$) only the second-order Coriolis contribution exists and the second-order non-spherically-symmetric distortion which results from centrifugal force vanishes for a uniform rotation (see Eq. (64)). 4) In non-radial cases, at lower radial orders, the Coriolis contribution and non-spherically-symmetric distortion have the same order of magnitude. At higher orders, however, the magnitude of the non-spherically-symmetric distortion becomes greater than Coriolis contribution. 5) With increasing n, the frequency correction due to distortion and Coriolis coupling increases and decreases respectively.

Table 2 Values of eigenfrequency σ_0 (including the $O(\Omega)$ contribution from rotation), second- and third-order Coriolis contributions σ_2^{T} and σ_3^{T}, second- and third-order non-spherically-symmetric distortions σ_2^{D} and σ_3^{D}, third-order distortion and Coriolis coupling σ_3^{C}, and total frequency $\sigma_{\mathrm{tot}} = \sigma_0 + \sigma_{\mathrm{c}}$ corrected up to third order, for p modes in a zero-age main-sequence star with $M = 12M_\odot$, $(l, m) = (1, 1)$ and $n = (1, \dots, 15)$. Here $\sigma_{\mathrm{c}} = \sigma_2^{\mathrm{T}} + \sigma_3^{\mathrm{T}} + \sigma_2^{\mathrm{D}} + \sigma_3^{\mathrm{D}} + \sigma_3^{\mathrm{C}}$ is the total third-order frequency correction. All frequencies are in units of $\sqrt{GM/R^3} = 2.42 \times 10^{-4}\,\mathrm{s}^{-1}$.

Mode	n	σ_0	σ_2^{T}	σ_2^{D}	σ_3^{T}	σ_3^{D}	σ_3^{C}	σ_{tot}
p_1	1	3.3700	2.940×10^{-3}	-1.873×10^{-3}	-1.43×10^{-4}	6.53×10^{-5}	6.98×10^{-5}	3.3711
p_2	2	4.5499	2.173×10^{-3}	-2.227×10^{-3}	-7.91×10^{-5}	5.85×10^{-5}	7.45×10^{-5}	4.5499
p_3	3	5.6388	1.759×10^{-3}	-2.695×10^{-3}	-5.27×10^{-5}	5.87×10^{-5}	9.54×10^{-5}	5.6380
p_4	4	6.6669	1.483×10^{-3}	-2.777×10^{-3}	-3.81×10^{-5}	5.22×10^{-5}	1.06×10^{-4}	6.6657
p_5	5	7.7061	1.285×10^{-3}	-2.701×10^{-3}	-2.88×10^{-5}	4.44×10^{-5}	1.01×10^{-4}	7.7048
p_6	6	8.8212	1.129×10^{-3}	-2.648×10^{-3}	-2.23×10^{-5}	3.85×10^{-5}	9.12×10^{-5}	8.8198
p_7	7	9.9704	1.003×10^{-3}	-2.522×10^{-3}	-1.77×10^{-5}	3.28×10^{-5}	8.32×10^{-5}	9.9690
p_8	8	11.152	8.986×10^{-4}	-2.051×10^{-3}	-1.42×10^{-5}	2.40×10^{-5}	7.87×10^{-5}	11.151
p_9	9	12.336	8.098×10^{-4}	-5.514×10^{-4}	-1.17×10^{-5}	5.88×10^{-6}	7.89×10^{-5}	12.336
p_{10}	10	13.503	7.323×10^{-4}	3.330×10^{-3}	-9.67×10^{-6}	-3.26×10^{-5}	8.44×10^{-5}	13.507
p_{11}	11	14.635	6.661×10^{-4}	1.135×10^{-2}	-8.15×10^{-6}	-1.03×10^{-4}	9.17×10^{-5}	14.647
p_{12}	12	15.741	6.153×10^{-4}	2.329×10^{-2}	-7.01×10^{-6}	-1.97×10^{-4}	9.26×10^{-5}	15.765
p_{13}	13	16.851	5.780×10^{-4}	3.771×10^{-2}	-6.17×10^{-6}	-2.99×10^{-4}	8.51×10^{-5}	16.889
p_{14}	14	17.971	5.472×10^{-4}	5.833×10^{-2}	-5.48×10^{-6}	-4.35×10^{-4}	7.58×10^{-5}	18.029
p_{15}	15	19.065	5.197×10^{-4}	1.110×10^{-1}	-4.92×10^{-6}	-7.83×10^{-4}	6.84×10^{-5}	19.176

In Figures 3–6 the results in Tables 2–5 are plotted. The plots show that for high-order p modes: 1) σ_0 increases as a linear function of n, which is very similar to the pattern of vibration in a simple string; this also follows from the asymptotic analyses of high-order acoustic modes (e.g., Vandakurov 1967; Tassoul 1980).

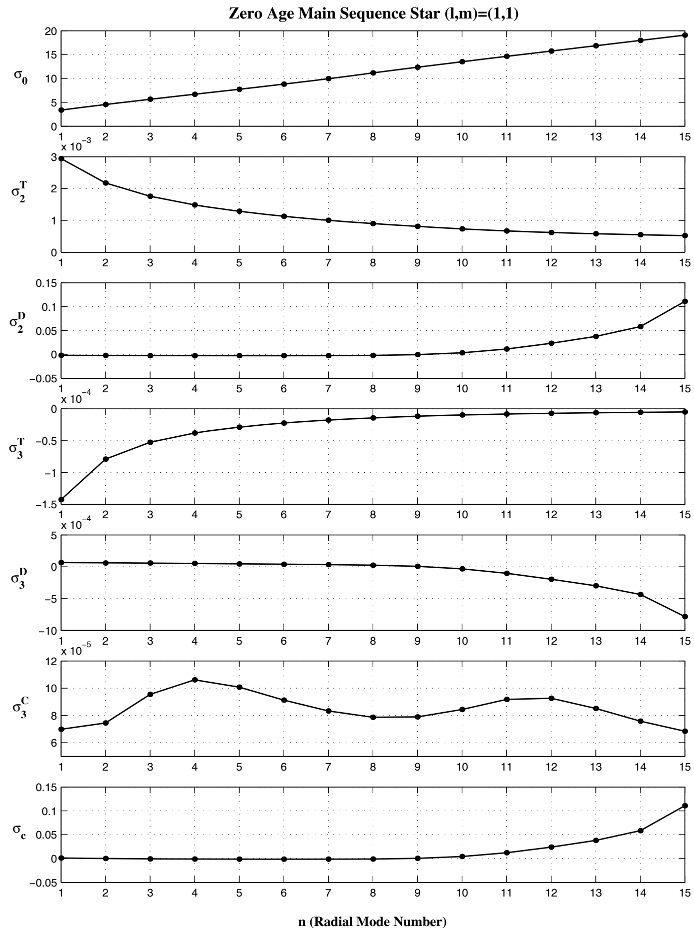

Fig. 3 Frequency corrections versus radial order n for p modes with $(l, m) = (1, 1)$ and $n = (1, \ldots, 15)$ for a zero-age main-sequence star with $M = 12M_\odot$. Here σ_0 is the zero-order eigenfrequency, σ_2^{T} and σ_3^{T} are the second- and the third-order Coriolis contributions, σ_2^{D} and σ_3^{D} are the second- and the third-order non-spherically-symmetric distortions, σ_3^{C} is the third-order distortion and Coriolis coupling, and $\sigma_{\mathrm{c}} = \sigma_2^{\mathrm{T}} + \sigma_3^{\mathrm{T}} + \sigma_2^{\mathrm{D}} + \sigma_3^{\mathrm{D}} + \sigma_3^{\mathrm{C}}$ is the total third-order frequency correction. All frequencies are in units of $\sqrt{GM/R^3} = 2.42 \times 10^{-4}\,\mathrm{s}^{-1}$.

2) (σ_2^{T}, σ_2^{D}) and (σ_3^{T}, σ_3^{D}) have almost regular asymptotic behaviors as $1/\sigma_0 \propto 1/n$ and $1/\sigma_0^2 \propto 1/n^2$, respectively. See Equations (52), (64) and (53), (65). 4) The distortion and Coriolis coupling, σ_3^{C}, has no clear asymptotic relation (see Eq. (69)). 5) The total frequency shift corrected up to third order, σ_{c}, for radial and non-radial modes decreases and increases, respectively, with increasing n; in the non-radial case this is caused by the dominance of the term σ_2^{D} which vanishes in the radial case. Note that for the case of $(l, m) = (2, 2)$, in the diagrams of σ_2^{T} and σ_3^{T} a break between f and p_1 appears. This happens because the inertia for the f mode is two orders of magnitude greater than for p_1.

In Table 6, the results on the third-order frequency corrections for the case of two near-degenerate modes, derived from Equations (75)–(76), are tabulated. As discussed above the coupling exists only for the two near-degenerate poloidal modes belonging to the same m but with l differing by ± 2.

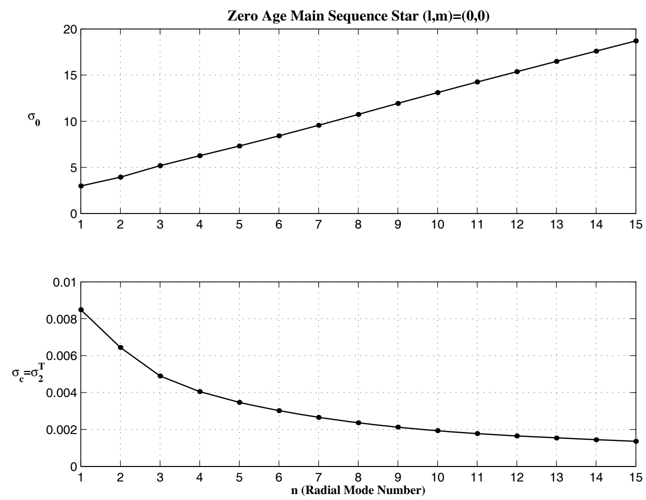

Fig. 4 Same as Fig. 3, for p modes with $(l, m) = (0, 0)$ and $n = (1, \ldots, 15)$. There are no σ_3^{T}, σ_3^{D}, and $\sigma_3^{\mathrm{C}} = 0$ for $m = 0$ (see Eqs. (53), (65) and (69)). Also σ_2^{D} vanishes since $l = m = 0$ and the model is a uniformly rotating star (see Eq. (64)).

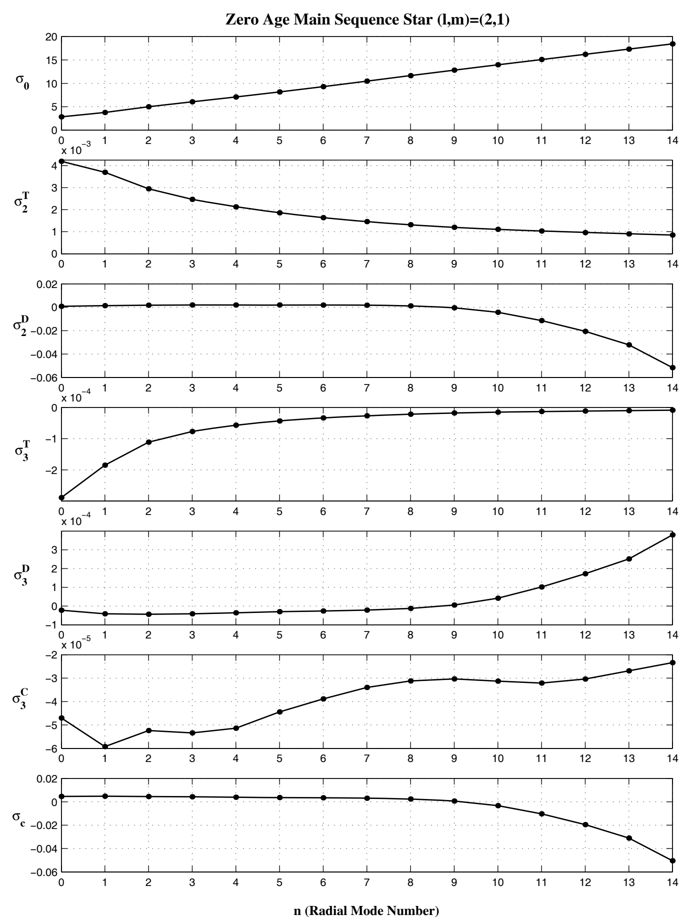

Fig. 5 Same as Fig. 3, for f and p modes with $(l, m) = (2, 1)$ and $n = (0, \ldots, 14)$.

Table 3 Same as Table 2, for p modes with $(l, m) = (0, 0)$ and $n = (1, \ldots, 15)$. There is no σ_3^{T}, σ_3^{D}, and $\sigma_3^{\mathrm{C}} = 0$ for $m = 0$ (see Eqs. (53), (65) and (69)). Also σ_2^{D} vanishes since $l = m = 0$ and the model is a uniformly rotating star (see Eq. (64)).

Mode	n	σ_0	σ_2^{T}	σ_{tot}
p_1	1	2.9941	8.488×10^{-3}	3.0026
p_2	2	3.9460	6.440×10^{-3}	3.9524
p_3	3	5.1853	4.901×10^{-3}	5.1902
p_4	4	6.2697	4.053×10^{-3}	6.2738
p_5	5	7.3206	3.471×10^{-3}	7.3241
p_6	6	8.4203	3.018×10^{-3}	8.4233
p_7	7	9.5645	2.657×10^{-3}	9.5671
p_8	8	10.742	2.366×10^{-3}	10.745
p_9	9	11.932	2.130×10^{-3}	11.934
p_{10}	10	13.112	1.938×10^{-3}	13.114
p_{11}	11	14.261	1.782×10^{-3}	14.262
p_{12}	12	15.374	1.653×10^{-3}	15.376
p_{13}	13	16.478	1.542×10^{-3}	16.479
p_{14}	14	17.591	1.445×10^{-3}	17.592
p_{15}	15	18.702	1.359×10^{-3}	18.704

Table 4 Same as Table 2, for f and p modes with $(l, m) = (2, 1)$ and $n = (0, \ldots, 14)$.

Mode	n	σ_0	σ_2^{T}	σ_2^{D}	σ_3^{T}	σ_3^{D}	σ_3^{C}	σ_{tot}
f	0	2.8330	4.196×10^{-3}	7.327×10^{-4}	-2.89×10^{-4}	-2.25×10^{-5}	-4.70×10^{-5}	2.8376
p_1	1	3.7712	3.696×10^{-3}	1.364×10^{-3}	-1.85×10^{-4}	-4.13×10^{-5}	-5.92×10^{-5}	3.7760
p_2	2	4.9925	2.947×10^{-3}	1.750×10^{-3}	-1.11×10^{-4}	-4.38×10^{-5}	-5.24×10^{-5}	4.9970
p_3	3	6.0592	2.470×10^{-3}	1.960×10^{-3}	-7.72×10^{-5}	-4.16×10^{-5}	-5.34×10^{-5}	6.0635
p_4	4	7.0920	2.129×10^{-3}	1.953×10^{-3}	-5.71×10^{-5}	-3.59×10^{-5}	-5.14×10^{-5}	7.0959
p_5	5	8.1662	1.856×10^{-3}	1.890×10^{-3}	-4.33×10^{-5}	-3.04×10^{-5}	-4.44×10^{-5}	8.1698
p_6	6	9.2940	1.638×10^{-3}	1.849×10^{-3}	-3.36×10^{-5}	-2.63×10^{-5}	-3.89×10^{-5}	9.2974
p_7	7	10.458	1.460×10^{-3}	1.697×10^{-3}	-2.67×10^{-5}	-2.15×10^{-5}	-3.40×10^{-5}	10.461
p_8	8	11.641	1.317×10^{-3}	1.138×10^{-3}	-2.16×10^{-5}	-1.30×10^{-5}	-3.12×10^{-5}	11.643
p_9	9	12.820	1.200×10^{-3}	-4.768×10^{-4}	-1.80×10^{-5}	4.99×10^{-6}	-3.03×10^{-5}	12.821
p_{10}	10	13.973	1.107×10^{-3}	-4.344×10^{-3}	-1.53×10^{-5}	4.18×10^{-5}	-3.13×10^{-5}	13.970
p_{11}	11	15.092	1.030×10^{-3}	-1.138×10^{-2}	-1.32×10^{-5}	1.02×10^{-4}	-3.21×10^{-5}	15.082
p_{12}	12	16.197	9.630×10^{-4}	-2.066×10^{-2}	-1.15×10^{-5}	1.72×10^{-4}	-3.04×10^{-5}	16.177
p_{13}	13	17.312	9.018×10^{-4}	-3.214×10^{-2}	-1.01×10^{-5}	2.51×10^{-4}	-2.69×10^{-5}	17.281
p_{14}	14	18.433	8.470×10^{-4}	-5.165×10^{-2}	-8.90×10^{-6}	3.80×10^{-4}	-2.34×10^{-5}	18.383

Table 5 Same as Table 2, for f and p modes with $(l, m) = (2, 2)$ and $n = (0, \ldots, 14)$.

Mode	n	σ_0	σ_2^{T}	σ_2^{D}	σ_3^{T}	σ_3^{D}	σ_3^{C}	σ_{tot}
f	0	2.7478	6.772×10^{-4}	-1.418×10^{-3}	-5.38×10^{-5}	8.90×10^{-5}	1.07×10^{-4}	2.7472
p_1	1	3.6591	1.204×10^{-3}	-2.159×10^{-3}	-8.78×10^{-5}	1.30×10^{-4}	1.73×10^{-4}	3.6584
p_2	2	4.8688	1.144×10^{-3}	-2.276×10^{-3}	-6.83×10^{-5}	1.14×10^{-4}	1.96×10^{-4}	4.8679
p_3	3	5.9318	9.859×10^{-4}	-1.848×10^{-3}	-4.97×10^{-5}	7.88×10^{-5}	2.30×10^{-4}	5.9312
p_4	4	6.9627	8.531×10^{-4}	-1.224×10^{-3}	-3.71×10^{-5}	4.51×10^{-5}	2.36×10^{-4}	6.9625
p_5	5	8.0358	7.468×10^{-4}	-9.157×10^{-4}	-2.83×10^{-5}	2.95×10^{-5}	2.10×10^{-4}	8.0359
p_6	6	9.1626	6.624×10^{-4}	-6.114×10^{-4}	-2.22×10^{-5}	1.74×10^{-5}	1.91×10^{-4}	9.1629
p_7	7	10.326	5.932×10^{-4}	-1.074×10^{-4}	-1.78×10^{-5}	2.74×10^{-6}	1.73×10^{-4}	10.326
p_8	8	11.508	5.350×10^{-4}	1.394×10^{-3}	-1.44×10^{-5}	-3.21×10^{-5}	1.66×10^{-4}	11.510
p_9	9	12.687	4.848×10^{-4}	5.227×10^{-3}	-1.19×10^{-5}	-1.10×10^{-4}	1.68×10^{-4}	12.693
p_{10}	10	13.840	4.406×10^{-4}	1.378×10^{-2}	-9.97×10^{-6}	-2.66×10^{-4}	1.81×10^{-4}	13.854
p_{11}	11	14.959	4.036×10^{-4}	2.844×10^{-2}	-8.48×10^{-6}	-5.10×10^{-4}	1.91×10^{-4}	14.987
p_{12}	12	16.064	3.759×10^{-4}	4.673×10^{-2}	-7.37×10^{-6}	-7.82×10^{-4}	1.84×10^{-4}	16.111
p_{13}	13	17.181	3.548×10^{-4}	6.849×10^{-2}	-6.51×10^{-6}	-1.07×10^{-3}	1.65×10^{-4}	17.249
p_{14}	14	18.305	3.364×10^{-4}	1.043×10^{-1}	-5.81×10^{-6}	-1.54×10^{-3}	1.47×10^{-4}	18.408

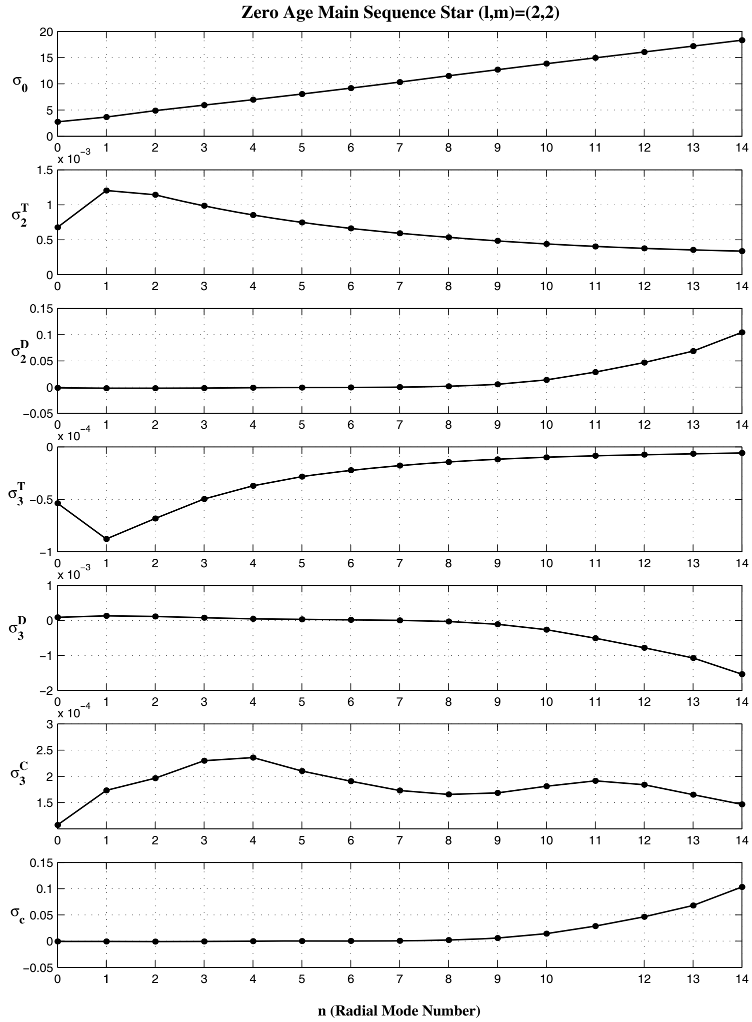

Fig. 6 Same as Fig. 3, for f and p modes with $(l, m) = (2, 2)$ and $n = (0, \dots, 14)$.

Table 6 Values of zero-order eigenfrequency σ_0, total frequency $\sigma_\pm$ corrected up to third order, total third-order frequency correction $\Delta\sigma_\pm = (\sigma_\pm - \sigma_0)$ due to rotation and coupling, expansion coefficients $\mathcal{A}_1^{(\pm)}$ and $\mathcal{A}_2^{(\pm)}$ normalized to 1, i.e., $\mathcal{A}_1^2 + \mathcal{A}_2^2 = 1$, for selected pairs of near-degenerate poloidal modes in a zero-age main-sequence star with $M = 12M_\odot$. All frequencies are in units of $\sqrt{GM/R^3} = 2.42 \times 10^{-4}\,\mathrm{s}^{-1}$.

m	l	n	coupling	σ_0	$\sigma_\pm$	$\Delta\sigma_\pm$	$\mathcal{A}_1^{(\pm)}$	$\mathcal{A}_2^{(\pm)}$
0	0	1	p_1	2.9941	3.0027	8.5940×10^{-3}	0.99950	−0.03152
0	2	0	f	2.9199	2.9262	6.2499×10^{-3}	0.04412	0.99903
0	0	3	p_3	5.1853	5.1907	5.3200×10^{-3}	0.99719	0.07496
0	2	2	p_2	5.1189	5.1266	7.6955×10^{-3}	0.08727	−0.99618
0	0	9	p_9	11.932	11.934	2.5947×10^{-3}	0.99858	0.05319
0	2	8	p_8	11.775	11.782	7.0084×10^{-3}	0.05738	−0.99835

6 CONCLUDING REMARKS

The third-order effect of rotation on the p and f modes for a uniformly rotating zero-age main-sequence star of mass $12M_\odot$ has been investigated. The third-order perturbation formalism presented by S98 was used, with corrections for some misprints and missing terms in some of their equations. Following S98, the Coriolis and spherically-symmetric distortion effects were included in the zero-order eigensystem. This yields eigenfrequencies ω_0 of eigenmodes which are no longer m-degenerate, even at zero order. Furthermore, this procedure enables one to obtain eigenfrequencies with the required ϵ^3 accuracy without the computation of eigenfunction corrections at successive orders of ϵ. The zero-order eigenvalue problem was solved by the pulsation code modified in this manner. Numerical calculations of the oscillation frequencies were carried out for our selected model and the second- and third-order frequency corrections due to Coriolis, non-spherically-symmetric distortion and Coriolis-distortion coupling were computed. For the case of $m = 0$ there are no first- and third-order frequency corrections. For the case of radial oscillation ($l = m = 0$) the second-order non-spherically-symmetric distortion is also zero and only the second-order Coriolis contribution exists. We discuss the validity of neglecting the surface terms which arise when the density derivatives are removed through an integration by parts. They become significant for higher-order modes, particularly in the present model whose atmosphere is relatively thin. Coupling only occurs between two poloidal modes with the same m and with l differing by 0 or 2.

We have carried out a careful comparison with the results of the independent implementation of the third-order formalism by S98. After taking into account the modifications discussed in Appendix A the results of the two formalisms for the combined second- and third-order corrections agree to within a few per cent.

In a subsequent paper we intend to investigate numerically the effect of rotation up to third-order for a sequence of β Cephei star models with uniform or radially varying rotation profiles.

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Appendix A: SUMMARY OF MODIFICATIONS COMPARED WITH THE FORMULATION OF S98

The third-order perturbation formalism for the influence of rotation on stellar oscillations that is presented here is to a large extent re-derivation of the results of S98. We consider it necessary to present some results of the derivations because of a number of typographical errors and some missing terms in S98, which might lead to confusion. For convenience we here summarize the required corrections to that paper, and other modifications in the present formulation:

- There are several corrections to S98, equations (A10). In the second of these equations the term $-h_1^2/\zeta$ is mistakenly omitted (cf. Eq. (21)). In the same equation $U/(1-\sigma_r)$ must be used instead of U in the radial case (cf. Eq. (35)). In the last of S98, equations (A10), the term in v should be

$$\left(\Lambda - \frac{UV_g}{1-\sigma_r}\right) v$$

(cf. Eq. (23)).
- In the definition of ζ, S98 equations (A11), C_r must be used instead of C (cf. Eq. (31)).
- In equation (A15), S98 neglected the factor $(g_e/\tilde{g})$ as being $O(\Omega^2)$ and approximated $(g_e/\tilde{g})v_k$ by v_k. We use the full expression in Equation (41).

– The factor $2J/\omega_0$ in equation (A22) of S98 was mistakenly omitted. The corrected form of the relation should be

$$\langle \xi_{\mathrm{t}1q} \mid \tilde{\rho}\xi_{\mathrm{t}1k} \rangle = \delta_{m_k,m_q} \left(\frac{2J_{kq}}{\omega_0}\right)\left(\frac{\bar{\Omega}}{\omega_0}\right)^2 [\delta_{l_k,l_q}K_1 + \delta_{l_k,l_q+2}K_{2kq} + \delta_{l_k,l_q-2}K^*_{2qk}] \ , \tag{A.1}$$

where K_1 and K_{2kq} are given by equation (B5) in S98.

A.1. Surface Effects

We have considered two approaches for the computation of the frequency corrections. In the first, the corrections are calculated from equations (B6)–(B7), (B19)–(B21), and (B25) of S98. In the second which is used in present work, those coefficients are computed from Equations (52), (53), (64), (65) and (69).

There are two substantial differences between the equations used in the two approaches. In the formulation of S98, the density derivatives are eliminated through an integration by parts and the resulting surface terms are ignored. For instance, in equation (B25) of S98 the neglected surface term is

$$\mathrm{S.T} = \frac{m}{J}\frac{\bar{\Omega}^3}{\omega_0}\, r^5 u_2(1+\eta)(\delta_{l_k l_q} g_1 + \mathcal{Q}_{kq2} g_2)\tilde{\rho}\big|_{r=R} \ , \tag{A.2}$$

where $g_1 = C\sigma_0^2 z_k z_q$ and $g_2 = C\sigma_0^2(y_k z_q + y_q z_k + 3z_k z_q)$. In Figure A1 the surface term given by Equation (A2) as well as its effect on the third-order distortion and Coriolis coupling correction, $\sigma^{\mathrm{C}}_{3,\mathrm{Soufi}}$ (see eq. (B25) in S98), are plotted against radial order. The plot shows that for lower radial orders n the effect of surface term is negligible, whereas for higher orders it becomes important. Note that for higher radial orders the dominant term in surface term comes mostly from g_2. Our study shows that for the model used in the present paper the surface terms become considerable. Because the boundary has been located at $X{:=}\ r/R = 1$, where the density is not exactly zero.

A.2. Using the Approximation $z \simeq y_t/C\sigma_0^2$ in the Third-Order Correction Terms

The other important difference which should be noted is that in the S98 approach for computing the third-order correction terms the approximation $z \simeq y_t/C\sigma_0^2$, which is valid for the non-rotating case, is used. In the present paper, on the other hand, the exact relation Equation (24) is used. To investigate this difference in detail, in Figure A2 the two terms included in Equation (24) as well as the full relation for z are plotted for several p modes with $(l, m) = (1, 1)$ and $n = (3, 8, 15)$ in the $12M_\odot$ ZAMS model considered here. Figure A2 shows that from the centre to near the surface the second term $h_1 y/\Lambda$ is negligible compared with the first term $\zeta y_t/\Lambda$. However, very close to the surface the second term becomes important and with increasing radial order n it becomes dominant compared with the first term. Hence, we conclude that the approximation of neglecting the second term everywhere, particularly near the surface, is not valid. We find that the magnitude of this difference between the two approaches is more significant than the magnitude of the difference due to of the surface terms.

If we include the surface terms in the S98 approach and use the approximation $z \simeq y_t/C\sigma_0^2$ in the present formulation the results of the two numerical approaches are in good agreement. For lower radial orders, the difference in the combined second- and third-order frequency correction σ_{c} is then smaller than one percent and increases slightly for higher orders.

Appendix B: DIFFERENT CONTRIBUTIONS OF NON-SPHERICALLY-SYMMETRIC DISTORTION

The dimensionless off-diagonal terms $\bar{D}_{kq}$, in Equation (62), are split into five separate integrals for convenience:

$$\bar{D}^{(1)}_{kq} = -\frac{\sigma^2_{\bar{\Omega}}}{2\bar{J}_{kq}}\int_0^1 dx\bar{\tilde{\rho}}x^4\Gamma_1\lambda_k\lambda_q u_2 \ , \tag{B.1}$$

$$\begin{aligned}\bar{D}^{(2)}_{kq} \ =\ & \frac{\sigma^2_{\bar{\Omega}}}{2\bar{J}_{kq}}\int_0^1 dx\bar{\tilde{\rho}}x^4(1-\sigma_r)\{b_2 - (A+V_g)u_2\} \\ & \times\ \{y_q w_k + w_q y_k + (\bar{\Lambda}-3)(z_q v_k + v_q z_k)\} \ ,\end{aligned} \tag{B.2}$$

$$\begin{aligned}\bar{D}^{(3)}_{kq} &= -\frac{\sigma^2_{\bar{\Omega}}}{2J_{kq}}\int_0^1 dx\bar{\bar{\rho}}x^4\Big\{\frac{u_2}{2}\times\\ &\times[z_q(\Lambda_q-\Lambda_k+6)(\lambda_k-(A+V_g)y_k)\\ &\quad+(\lambda_q-(A+V_g)y_q)(\Lambda_k-\Lambda_q+6)z_k]\\ &+[b_2-(A+V_g)u_2+\tfrac{2}{3}(1+\eta_2)+\frac{1}{x}\frac{d\bar{\phi}_{22}}{dx}]\\ &\quad\times[\lambda_k y_q+y_k\lambda_q-2(A+V_g)y_k y_q]\Big\}\ ,\end{aligned} \tag{B.3}$$

$$\begin{aligned}\bar{D}^{(4)}_{kq} &= \frac{\sigma^2_{\bar{\Omega}}}{2J_{kq}}\int_0^1 dx\bar{\bar{\rho}}x^4\,\{(1-\sigma_r)\times\\ &\times[b_2-(A+V_g)u_2][(U+\chi-4)y_k y_q\\ &\quad+(\lambda_q+\Lambda_q z_q)y_k+(\lambda_k+\Lambda_k z_k)y_q]\\ &-y_k y_q(A+V_g)[(2-\sigma_r)(b_2-(A+V_g)u_2)\\ &\quad+\tfrac{2}{3}(1+\eta_2)+\frac{1}{x}\frac{d\bar{\phi}_{22}}{dx}]\\ &-2b_2[y_k y_q+\tfrac{1}{4}y_k z_q(\Lambda_q-\Lambda_k+6)\\ &\quad+\tfrac{1}{4}y_q z_k(\Lambda_k-\Lambda_q+6)]\}\ ,\end{aligned} \tag{B.4}$$

$$\begin{aligned}\bar{D}^{(5)}_{kq} &= -\frac{\sigma^2_{\bar{\Omega}}}{2J_{kq}}\int_0^1 dx\bar{\bar{\rho}}x^4 C(\frac{\sigma^2_{0k}+\sigma^2_{0q}}{2})\times\\ &\times[b_2-(A+V_g)u_2][y_k y_q+(\bar{\Lambda}-3)z_k z_q]\ ,\end{aligned} \tag{B.5}$$

where $\bar{\Lambda}=(\Lambda_k+\Lambda_q)/2$, and

$$\begin{aligned}\mathcal{Q}_{kk2} &= \tfrac{3}{2}(\beta^2_{k+1}+\beta^2_k)-\tfrac{1}{2}=(l_k+1)\beta^2_k-l_k\beta^2_{k+1}\\ &= \frac{\Lambda_k-3m_k^2}{4\Lambda_k-3}\ .\end{aligned} \tag{B.6}$$

Appendix C: THE HERMITICITY OF AN OSCILLATING ROTATING SYSTEM

The hermiticity of the equations of oscillation in a rotating fluid has already been investigated by Gough & Thompson (1990) who ensured the hermiticity for the rotating star by means of mapping each point

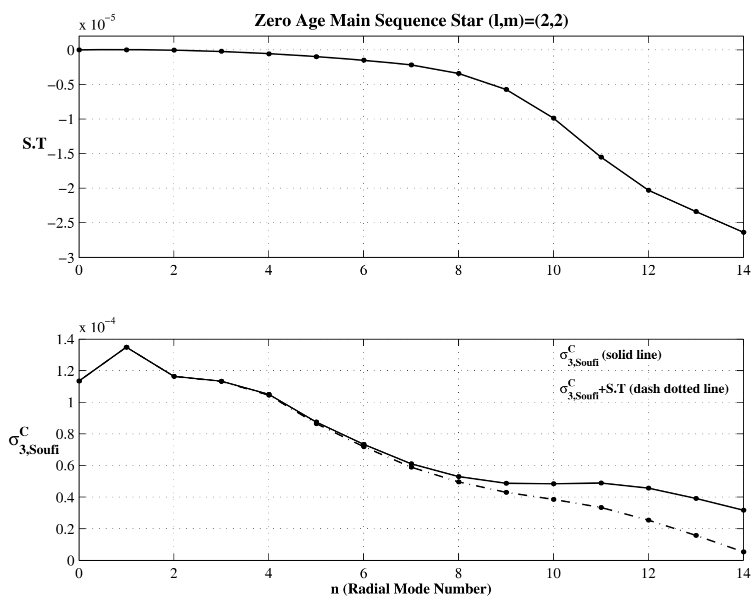

Fig. A.1 Surface term and its effect on eq. (B25) in Soufi et al. (1998) for p modes with $(l,m)=(2,2)$ and $n=(0,\ldots,14)$ for a zero-age main-sequence model.

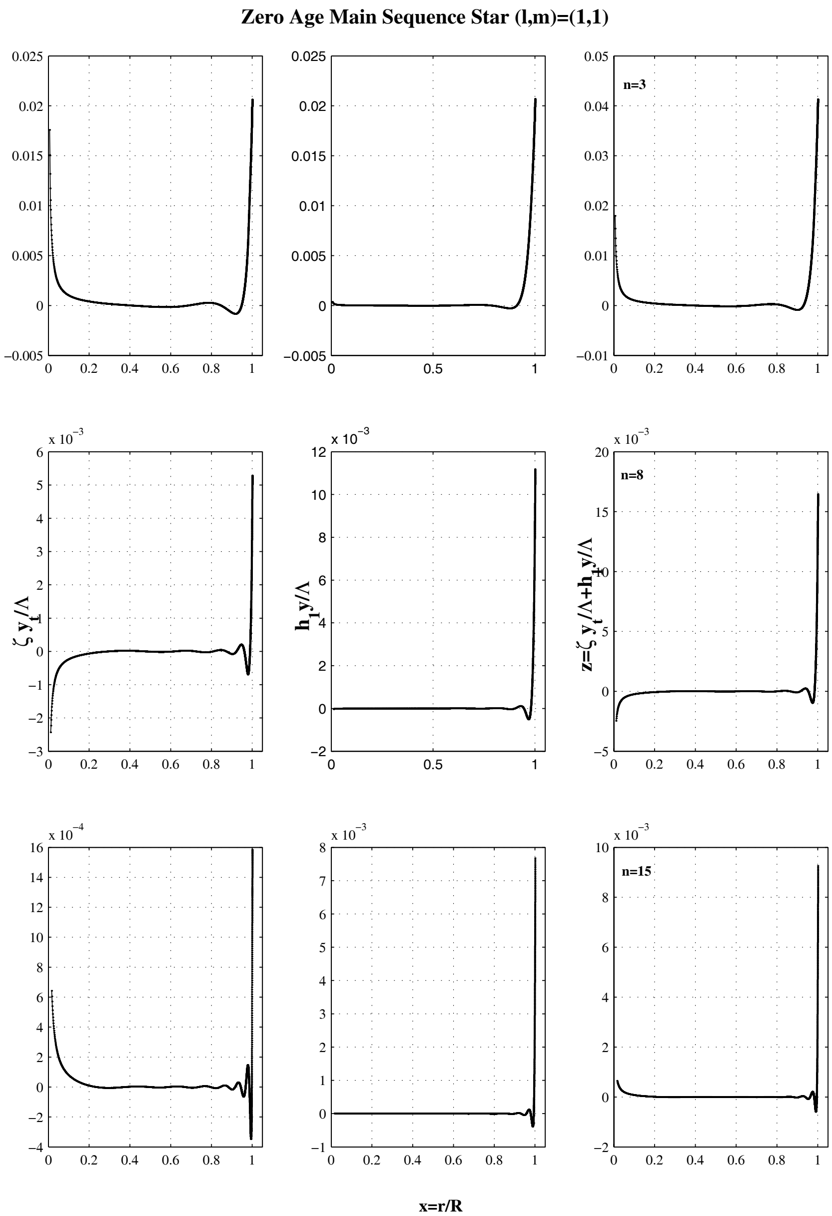

Fig. A.2 First $\zeta y_t/\Lambda$ (left), second $h_1 y/\Lambda$ (middle), and total terms (right) of zero-order horizontal component z of eigenfunction against fractional radius $x = r/R$ for selected p modes $(l, m) = (1, 1)$ and $n = (3, 8, 15)$.

in the distorted model to a corresponding point in the spherically symmetric stellar model. Lynden-Bell & Ostriker (1967) demonstrated that all of the linearized operators appearing in a rotating system for a constant rotation profile are Hermitian. Here we extend the argument of Lynden-Bell & Ostriker (1967) to the case of a rotation profile that is a function of radius.

To study the hermiticity of the operator $\mathcal{L}$, equation (21) in S98, we need to show only that the operator A, equation (22) in S98, is Hermitian. The other operators B, D and C which appear in equation (22) of

S98, also appear in the total coupling coefficient $\mathcal{H}$ (see eqs. (B1)–(B2) of S98), which is symmetric and Hermitian, as is shown in Section 4.

Let ξ and ζ be two displacement eigenvectors of an oscillating rotating system. Then one can show that the operator A, equation (22) of S98, satisfies the following relation

$$\langle \zeta \mid A\xi \rangle = \langle A\zeta \mid \xi \rangle + \mathrm{S.T.}\ , \tag{C.1}$$

where the surface term (S.T.) is

$$\begin{aligned} \mathrm{S.T.} = \oint_S \ & \{\tilde{p}'\zeta^* + (\zeta^* \cdot \nabla\tilde{p})\xi + (\Gamma_1\tilde{p}\nabla\cdot\zeta^*)\xi \\ & +\tilde{\rho}(\tilde{\phi}'\zeta^* - \hat{\phi}'^*\xi) \\ & + \frac{1}{4\pi G}(\tilde{\phi}'\nabla\hat{\phi}'^* - \hat{\phi}'^*\nabla\tilde{\phi}')\Big\} \cdot d\boldsymbol{S}\ , \end{aligned} \tag{C.2}$$

with $d\boldsymbol{S} = \boldsymbol{e}_r R^2 \sin\theta d\theta d\phi$; here hat and tilde indicate components of the eigenfunctions associated with ζ and ξ, respectively. In particular, $\hat{\phi}'$ is given by the Poisson equation, Equation (12), for eigenvector ζ. Following Unno et al. (1989), the relevant boundary conditions for an oscillating rotating star are

$$\tilde{p} = \tilde{\rho} = 0\,, \qquad \mathrm{at}\ r = R\,, \tag{C.3}$$

$$\tilde{p}' = 0\,, \qquad \mathrm{at}\ r = R\,, \tag{C.4}$$

$$\frac{d\phi'}{dr} + \frac{(l+1)}{r}\phi' = 0\,, \qquad \mathrm{at}\ r = R\,. \tag{C.5}$$

The first and second boundary conditions, Equations (C.3) and (C.4), result in removing all terms in Equation (C.2) that include $\tilde{p}$ and $\tilde{p}'$, and also all terms that include $\nabla\tilde{p}$ because they are proportional to $\tilde{\rho}$ (see eqs. (12) and (19) in S98). The last boundary condition eliminates the last surface term in Equation (C.2). Hence all the surface terms go to zero at the surface of the star and therefore $\mathrm{S.T.} = 0$. With the result Equation (C.1) becomes

$$\langle \zeta \mid A\xi \rangle = \langle A\zeta \mid \xi \rangle\ , \tag{C.6}$$

and therefore the operator A is Hermitian.

We note that from a theoretical point of view, the density vanishes exactly at the surface, whereas for a realistic model obtained from numerical calculations the density is not exactly zero at the surface. Hence from a numerical point of view the surface terms are not exactly zero; consequently, whether the surface term can be removed or not depends on the required accuracy. For instance, we saw before that the effect of surface terms is not negligible when the cubic frequency corrections are computed.

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